Cross-diffusion systems with entropy structure

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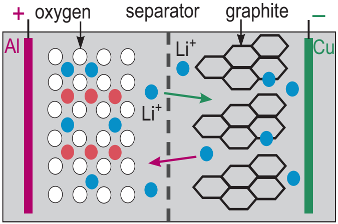


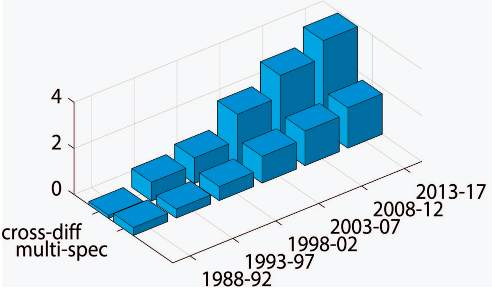
Multi-species systems

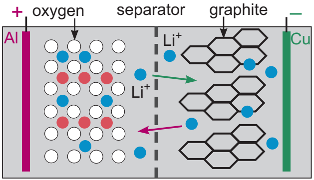
Examples:

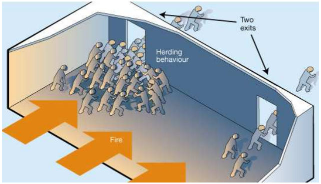
- Wildlife populations
- Tumor growth
- Gas mixtures
- Lithium-ion batteries
- Population herding

Nature is composed of multi-species systems

Relative number of publications (in 0.01%):







Modeling of multi-species systems

- Particle models: Newton's laws with interactions among species
- Markov chains: species move to neighboring cells
- Stochastic differential equations: using Brownian motion
- Kinetic equations: distribution function depends on age, size, etc.
- Here: Diffusive equations for population densities

Reaction-diffusion systems:

$$\partial_t u_i - \mathrm{div}(D_i \nabla u_i) = f_i(u) \text{ in } \Omega,\ t > 0, \quad u_i(0) = u_i^0, \quad \text{no-flux b.c.}$$

- Flux $D_i \nabla u_i$ only depends on u_i: Fick's law not always valid!
- In multicomponent systems, flux may depend on $\nabla u_1, \ldots, \nabla u_n$

Cross-diffusion systems:

$$\partial_t u - \mathrm{div}(A(u)\nabla u) = f(u) \text{ in } \Omega,\ t > 0, \quad u(0) = u^0, \quad \text{no-flux b.c.}$$

- Meaning: $\mathrm{div}(A(u)\nabla u)_i = \sum_{j=1}^n \mathrm{div}(A_{ij}(u)\nabla u_j)$, $A \in \mathbb{R}^{n\times n}$, $u \in \mathbb{R}^n$
- Cross-diffusion may allow for pattern formation

Example ❶: Cross-diffusion population dynamics

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u) \text{ in } \Omega,\ t > 0, \quad u(0) = u^0, \quad \text{no-flux b.c.}$$

- $u = (u_1, u_2)$ and u_i models population density of ith species
- Diffusion matrix: $(a_{ij} \geq 0)$

$$A(u) = \begin{pmatrix} a_{10} + a_{11}u_1 + a_{12}u_2 & a_{12}u_1 \\ a_{21}u_2 & a_{20} + a_{21}u_1 + a_{22}u_2 \end{pmatrix}$$

- Suggested by Shigesada-Kawasaki-Teramoto 1979 to model segregation
- Derivation from on-lattice model
- Lotka-Volterra functions: $f_i(u) = (b_{i0} - b_{i1}u_1 - b_{i2}u_2)u_i$
- Diffusion matrix is not symmetric, generally not positive definite

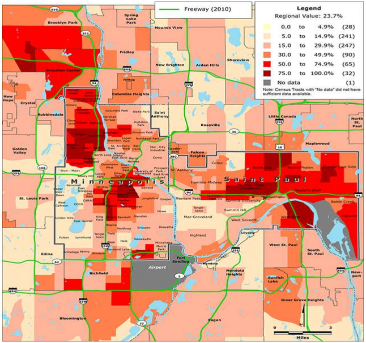


Figure: Minneapolis-Saint Paul percentage minority population 2010

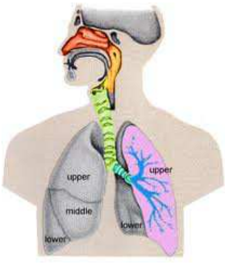
Example ❷: Multicomponent gas mixtures

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u) \text{ in } \Omega,\ t > 0, \quad u(0) = u^0, \quad \text{no-flux b.c.}$$

- Volume fractions of gas components $u_1, \ldots, u_n$, $u_{n+1} = 1 - \sum_{i=1}^n u_i$
- Diffusion matrix for $n = 2$: $\delta(u) = d_1 d_2(1 - u_1 - u_2) + d_0(d_1 u_1 + d_2 u_2)$

$$A(u) = \frac{1}{\delta(u)} \begin{pmatrix} d_2 + (d_0 - d_2)u_1 & (d_0 - d_1)u_1 \\ (d_0 - d_2)u_2 & d_1 + (d_0 - d_1)u_2 \end{pmatrix}$$

- Application: Patients with airways obstruction inhale Heliox to speed up diffusion
- Proposed by Maxwell 1866/Stefan 1871
- Duncan-Toor 1962: Fick's law ($J_i \sim \nabla u_i$) not sufficient, include cross-diffusion terms
- Boudin-Grec-Salvarani 2015: Derivation from Boltzmann equation for simple mixtures



Difficulties and objectives

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u) \text{ in } \Omega,\ t > 0, \quad u(0) = u^0$$

Main features:

- Diffusion matrix $A(u)$ non-diagonal (cross-diffusion)
- Matrix $A(u)$ may be neither symmetric nor positive definite
- Variables u_i expected to be bounded from below and/or above

Objectives:

- Local-in-time existence and uniqueness of classical solutions
- Global-in-time existence and uniqueness of weak solutions
- Positivity and boundedness of solution (if physically expected)
- Large-time behavior, design of stable numerical schemes

Mathematical difficulties:

- No general theory for diffusion systems
- Generally no maximum principle, no regularity theory
- Lack of positive definiteness $\Rightarrow$ local/global existence nontrivial

Overview

Local existence analysis

$$\partial_t u - \mathrm{div}(A(u)\nabla u) = f(u) \text{ in } \Omega \subset \mathbb{R}^d,\ \ t > 0, \quad u(0) = u^0$$

Theorem (Amann 1990)

Let a_{ij}, f_i smooth, $A(u)$ normally elliptic, $u^0 \in W^{1,p}(\Omega;\mathbb{R}^n)$ with $p > d$. Then $\exists$ unique local solution u

$$u \in C^0([0,T^*);W^{1,p}(\Omega)), \quad u \in C^\infty(\overline{\Omega}\times[0,T^*);\mathbb{R}^n), \quad 0 < T^* \le \infty$$

- $A(u)$ normally elliptic = all eigenvalues have positive real parts
- Linear algebra: If $H(u)$ symmetric positive definite such that $H(u)A(u)$ positive definite then $A(u)$ normally elliptic
- Application: Let $h(u)$ convex and set $H(u) := h''(u)$. Then, if $f = 0$,

$$\frac{d}{dt}\int_\Omega h(u)dx = \int_\Omega h'(u)\cdot\partial_t u dx = -\int_\Omega \underbrace{\nabla u : h''(u)A(u)\nabla u}_{\ge 0 \text{ if } h''(u)A(u) \text{ pos. def.}} dx$$

- Aim: find a Lyapunov functional (entropy) $\int_\Omega h(u)dx$

State of the art

$$\partial_t u - \mathrm{div}(A(u)\nabla u) = f(u) \text{ in } \Omega \subset \mathbb{R}^d,\ \ t > 0$$

Global existence if ...

- Growth conditions on nonlinearities (Ladyženskaya ... 1988)
- Control on $W^{1,p}(\Omega)$ norm with $p > d$ (Amann 1989)
- Positivity, mass control, diagonal $A(u)$ (Pierre-Schmitt 1997)

Unexpected behavior:

- Finite-time blow-up of Hölder solutions (Stará-John 1995)
- Weak solutions may exist after L^∞ blow-up (Pierre 2003)
- Cross-diffusion may lead to pattern formation (instability) or may avoid finite-time blow-up (Hittmeir-A.J. 2011)

Special structure needed for global existence theory:
gradient-flow or entropy structure

Entropy and gradient flows

Entropy: Measure of molecular disorder or energy dispersal

- Introduced by Clausius (1865) in thermodynamics
- Boltzmann, Gibbs, Maxwell: statistical interpretation
- Shannon (1948): concept of information entropy

Entropy in mathematics: $\sim$ convex Lyapunov functional

- Hyperbolic conservation laws (Lax), kinetic theory (Lions)
- Relations to stochastic processes (Bakry, Emery) and optimal transportation (Carrillo, Otto, Villani)

Gradient flow: $\partial_t u = -\mathrm{grad} H|_u$ on differential manifold

- Example: $\mathbb{R}^d$ with Euclidean structure $\Rightarrow \partial_t u = -H'(u)$
  $H(u)$ is Lyapunov functional since $\partial_t H(u) = -|H'(u)|^2$
- Gradient flow of entropy w.r.t. Wasserstein distance (Otto), entropy $H(u) = \int u \log u dx$: $\partial_t u = \mathrm{div}(u \nabla H'(u)) = \Delta u$

Gradient flows: Cross-diffusion systems

Main assumption

$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u)$ possesses formal gradient-flow structure

$$\partial_t u - \operatorname{div}\big(B\nabla \operatorname{grad} H(u)\big) = f(u),$$

where B is positive semi-definite, $H(u) = \int_\Omega h(u)dx$ entropy

Equivalent formulation: $\operatorname{grad} H(u) \simeq h'(u) =: w$ (entropy variable)

$$\partial_t u - \operatorname{div}(B\nabla w) = f(u), \quad B = A(u)h''(u)^{-1}$$

Consequences:

1. H is Lyapunov functional if $f = 0$:

$$\frac{dH}{dt} = \int_\Omega \partial_t u \cdot \underbrace{h'(u)}_{=w} dx = -\int_\Omega \nabla w : B\nabla w dx \leq 0$$

2. L^∞ bounds for u: Let $h' : D \to \mathbb{R}^n$ ($D \subset \mathbb{R}^n$) be invertible $\Rightarrow$ $u = (h')^{-1}(w) \in D$ (no maximum principle needed!)

Example: Maxwell-Stefan systems for $n = 2$

Volume fractions of gas components u_1, u_2, $u_3 = 1 - u_1 - u_2$

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = 0 \text{ in } \Omega,\ t > 0, \quad u(0) = u^0, \quad \text{no-flux b.c.}$$

$$A(u) = \frac{1}{\delta(u)} \begin{pmatrix} d_2 + (d_0 - d_2)u_1 & (d_0 - d_1)u_1 \\ (d_0 - d_2)u_2 & d_1 + (d_0 - d_1)u_2 \end{pmatrix}$$

Entropy: $H(u) = \int_\Omega h(u)dx$, where

$$h(u) = u_1(\log u_1 - 1) + u_2(\log u_2 - 1) + (1 - u_1 - u_2)(\log(1 - u_1 - u_2) - 1)$$

- Entropy variables: $w = h'(u) \in \mathbb{R}^2$ or $u = (h')^{-1}(w)$

$$w_i = \frac{\partial h}{\partial u_i} = \log \frac{u_i}{u_3}, \quad u_i = \frac{e^{w_i}}{1 + e^{w_1} + e^{w_2}} \in (0, 1)$$

- Entropy production:

$$\frac{dH}{dt}(u) = -\int_\Omega \left(\sum_{i=1}^{2} d_i \frac{|\nabla u_i|^2}{u_i} + d_0 u_1 u_2 \frac{|\nabla u_3|^2}{u_3} \right) dx \le 0$$

Overview

Boundedness-by-entropy method

$\partial_t u(w) - \operatorname{div}(B(w)\nabla w) = f(u(w))$ in Ω, $t > 0$, $\quad u|_{t=0} = u^0$, $\quad$ no-flux b.c.

$$\frac{d}{dt}\int_\Omega h(u)dx = -\int_\Omega \nabla u : h''(u)A(u)\nabla u dx + \int_\Omega f(u)\cdot h'(u)dx$$

Assumptions:

1. $\exists$ entropy density $h \in C^2(D;[0,\infty))$, h' invertible on $D \subset \mathbb{R}^n$
   Example: $h(u) = u\log u$ for $u \in D = (0,\infty)$,
   $u = (h')^{-1}(w) = e^w \in D$
2. "Degenerate" positive definiteness: $h''(u)A(u) \geq \operatorname{diag}(a_i(u_i)^2)$
   $$\nabla u : h''(u)A(u)\nabla u \geq \sum_{i=1}^n a_i(u_i)^2|\nabla u_i|^2$$
   Gives estimate for $|\nabla(u_i)^{m_i}|^2$ if $a_i(u_i) \sim u_i^{m_i-1}$
3. A continuous on D, $\exists C > 0 : \forall u \in D$: $f(u)\cdot h'(u) \leq C(1 + h(u))$
   Needed to control reaction term $f(u)$

Boundedness-by-entropy method

$$\partial_t u - \mathrm{div}(A(u)\nabla u) = f(u) \text{ in } \Omega,\ t > 0, \quad u(0) = u^0, \quad \text{no-flux b.c.}$$

Assumptions:

1. $\exists$ convex entropy $h \in C^2(D;[0,\infty))$, h' invertible on $D \subset \mathbb{R}^n$
2. "Degenerate" positive definiteness: for all $u \in D$,
$$z : h''(u)A(u)z \geq \sum_{i=1}^{n} a_i(u)^2 z_i^2, \quad a_i(u) \sim u_i^{m_i-1}$$
3. A continuous on D, $\exists C > 0 : \forall u \in D$: $f(u) \cdot h'(u) \leq C(1 + h(u))$

Theorem (A.J., *Nonlinearity* 2015)

Let the above assumptions hold, let $D \subset \mathbb{R}^n$ be bounded, $u^0 \in L^1(\Omega) \cap \overline{D}$. Then $\exists$ global weak solution such that $u(x,t) \in \overline{D}$ and
$$u \in L^2_{\mathrm{loc}}(0,\infty; H^1(\Omega)), \quad \partial_t u \in L^2_{\mathrm{loc}}(0,\infty; H^1(\Omega)')$$

Boundedness-by-entropy method

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Remarks:

- Result valid for rather general model class
- Yields L^∞ bounds without using a maximum principle
- Yields immediately global existence for Maxwell-Stefan systems $n = 2$
- Boundedness assumption on D is strong but can be weakened in some cases; see example below
- How to find entropy functions h? Physical intuition, trial and error

Ideas of proof

- Approximation: solve elliptic problem for w^k and $u^k = u(w^k)$

$$\frac{1}{\tau}(u^k - u^{k-1}) - \operatorname{div}\left(B(w^k)\nabla w^k\right) + \text{“}\varepsilon((-\Delta)^s w^k + w^k)\text{”} = f(u^k)$$

gives solutions $w^k \in H^s(\Omega) \subset L^\infty(\Omega)$ if $s > d/2$, $s \in \mathbb{N}$

- A priori estimate from entropy inequality

$$\frac{dH^k}{dt} + \int_\Omega \nabla u^k : h''(u^k)A(u^k)\nabla u^k dx \le C(1 + H^k)$$

gives uniform bounds for $\nabla(u_i^k)^{m_i}$ and $(u_i^k - u_i^{k-1})/\tau$

- Aubin-Lions lemma (Chen/A.J./Liu 2014): Let $u_i^{(\tau)}$ be piecewise constant in time with values u_i^k, $m_i \ge \frac{1}{2}$, and

$$\tau^{-1}\|u_i^{(\tau)}(t) - u_i^{(\tau)}(t-\tau)\|_{L^1(\tau,T;(H^k)')} + \|(u_i^{(\tau)})^{m_i}\|_{L^2(0,T;H^1)} \le C$$

Then $\exists$ subsequence $u_i^{(\tau)} \to u_i$ strongly in $L^{2m_i}(0,T;L^{2m_i})$

- Limit $(\varepsilon, \tau) \to 0$

❷ *n*-species Maxwell-Stefan equations

$$\partial_t u_i - \operatorname{div} J_i = f_i(u), \quad \nabla u_i = \sum_{j\neq i} c_{ij}(u_j J_i - u_i J_j) =: (CJ)_i$$

$$u_i(0) = u_i^0, \quad i = 1, \ldots, n, \quad \text{no-flux b.c.}$$

- Volume fractions u_i, fluxes J_i
- Problem: need to invert relation $\nabla u_i \leftrightarrow J_i$ but not invertible since $\sum_{i=1}^n u_i = 1 \Rightarrow \sum_{i=1}^n \nabla u_i = 0$
- Solution: solve $\nabla u = CJ$ on $\ker(C)^\perp \Rightarrow J^* = C_0^{-1}\nabla u^*$, where $u^* = (u_1, \ldots, u_{n-1})$, $J^* = (J_1, \ldots, J_{n-1})$

Entropy structure: $h(u^*) = \sum_{i=1}^n u_i(\log u_i - 1)$, $u_n = 1 - \sum_{i=1}^{n-1} u_i$

- Equations: $\partial_t u^* - \operatorname{div}(B(w)\nabla w) = f^*(u^*(w))$
- Difficulty: show that $B(w) = C_0^{-1} h''(u^*(w))^{-1}$ positive definite
- Boundedness-by-entropy theorem applies with $D = (0,1)^{n-1}$: $\exists$ global weak solution with

$$u_i^{1/2} \in L^2(0,T;H^1), \quad 0 \le u_i \le 1, \quad \sum_{i=1}^{n-1} u_i \le 1$$

❶ Population model of Shigesada-Kawasaki-Teramoto

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = 0 \text{ in } \Omega,\ t > 0, \quad u(0) = u^0, \quad \text{no-flux b.c.}$$

$$A(u) = \begin{pmatrix} a_{10} + a_{11}u_1 + a_{12}u_2 & a_{12}u_1 \\ a_{21}u_2 & a_{20} + a_{21}u_1 + a_{22}u_2 \end{pmatrix}$$

- Entropy: $H(u) = \int_\Omega h(u)dx = \int_\Omega \sum_{i=1}^2 u_i(\log u_i - 1)$ defined on unbounded domain $D = (0,\infty)^2$
- Entropy production: for some $C > 0$, if $f(u)$ = Lotka-Volterra term

$$\frac{dH}{dt}[u] \le -C\sum_{i=1}^2 \int_\Omega \big((a_{i0} + a_{ii}u_1)|\nabla\sqrt{u_1}|^2 + |\nabla\sqrt{u_1u_2}|^2\big)dx + C$$

- Main difficulty: We do not have (u_i) bounded in $L^\infty(\Omega)$ but only $(\sqrt{u_i})$ bounded in $L^6(\Omega)$ (if space dimension ≤ 3)

Theorem (Chen-A.J., *SIMA* 2004-2006)

Let $a_{i0} > 0$ or $a_{ii} > 0$. Then $\exists$ nonnegative weak solution (u_1, u_2)

Generalization 1: nonlinear coefficients

Macroscopic limit of random-walk on lattice with transition rates $p_i(u)$:

$$A(u) = \begin{pmatrix} p_1(u) + u_1 \frac{\partial p_1}{\partial u_1}(u) & u_1 \frac{\partial p_1}{\partial u_2}(u) \\ u_2 \frac{\partial p_2}{\partial u_1}(u) & p_2(u) + u_2 \frac{\partial p_2}{\partial u_2}(u) \end{pmatrix}$$

- p_i linear: Chen-A.J. 2004
- p_i sublinear: Desvillettes-Lepoutre-Moussa 2014
- p_i superlinear: $p_i(u) = a_{i0} + a_{i1} u_1^s + a_{i2} u_2^s$ $(i = 1, 2)$, entropy density: $h(u) = a_{21} u_1^s + a_{12} u_2^s$, $s > 1$

Theorem (A.J., *Nonlinearity* 2015)

Let $1 < s < 4$ and $(1 - \frac{1}{s}) a_{12} a_{21} \le a_{11} a_{22}$, $H(u^0) < \infty$.
Then $\exists$ nonnegative weak solution $u_i^{s/2} \in L^2_{\rm loc}(0, \infty; H^1(\Omega))$

- p_i superlinear, $s > 1$: Desvillettes-Lepoutre-Moussa-Trescases 2015

Generalization 2: more than two species

$$A_{ij}(u) = (a_{i0} + a_{i1}u_1 + \cdots + a_{in}u_n)\delta_{ij} + a_{ij}u_i$$

- Entropy: $H(u) = \int_\Omega h(u)dx = \int_\Omega \sum_{i=1}^n \pi_i u_i(\log u_i - 1)$
- Key assumption: $\pi_i a_{ij} = \pi_j a_{ji}$ (detailed balance), $\pi_i > 0$

Why detailed balance?

- Detailed balance $\Leftrightarrow$ (π_i) reversible measure $\Leftrightarrow$ $h''(u)A(u)$ symmetric $\Rightarrow$ entropy $H(u(t))$ decreases $\forall t$
- Detailed balance not satisfied: a_{ii} "large" $\Rightarrow$ $H(u(t))$ decreases, otherwise $\exists$ $u(0)$ such that $H(u(t))$ increases

Theorem (X. Chen-Daus-A.J. 2016)

Let $a_{ij} > 0$ and detailed balance hold. Then $\exists$ nonnegative weak solution $u_i^{1/2} \in L^2_{\rm loc}(0,\infty; H^1(\Omega))$, $i = 1,\ldots,n$

Nonlinear coefficients: Chen-Daus-A.J. 2016, Lepoutre-Moussa 2017

Further consequences

$$\partial_t u - \mathrm{div}(A(u)\nabla u) = f(u) \text{ in } \Omega,\ t > 0, \quad u(0) = u^0, \quad \text{no-flux b.c.}$$

Entropy structure: $H(u) = \int_\Omega h(u)dx$

$$\frac{dH}{dt} + \int_\Omega \nabla u : h''(u)A(u)\nabla u dx = \int_\Omega f(u) \cdot h'(u)dx$$

- Large-time asymptotics: left integral $\geq \kappa H$, $\kappa > 0$, right integral ≤ 0

$$\frac{dH}{dt} + \kappa H \leq 0 \quad \Rightarrow \quad H(t) \leq H(0)e^{-\kappa t}, \quad t \geq 0$$

- Uniqueness of weak solutions (Gajewski 1994): use semimetric

$$d(u,v) = \int_\Omega \Big(h(u) + h(v) - 2h\Big(\frac{u+v}{2}\Big)\Big)dx,\ h(u) = \sum_{i=1}^n u_i(\log u_i - 1)$$

and show that $\partial_t d(u,v) \leq 0$, $d(u(0),v(0)) = 0 \Rightarrow u(t) = v(t)$

Question: Often $h(u) = \sum_{i=1}^n u_i(\log u_i - 1)$. Are there other entropies?

Overview

Partial averaging in economics

- Reference: talk of P. L. Lions (Vienna 2015)
- Forward Kolmogorov equation with volatility $\sigma = \mathrm{diag}(\sigma_i)$, zero drift

$$\partial_t f = \frac{1}{2}\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}(\sigma_j^2 f) \quad \text{in } \mathbb{R}^n,\ t > 0$$

$f(x_1, \ldots, x_n, t)$ is probability density of Itô process

- Assumption: σ_j is function of partial averages

$$u_i(x,t) = \int_{\mathbb{R}} f(x, x_n, t) e^{\lambda_i x_n} dx_n, \quad x = (x_1, \ldots, x_{n-1})$$

- Interpretation: u_i = average with respect to economic parameter x_n
- Simplify: $i = 1, 2$, $\sigma = \sigma_j$, $\mu_i := \lambda_i^2 \sigma_n / 2$:

$$\partial_t u_i = \frac{1}{2}\Delta\big(\sigma(u)^2 u_i\big) + \mu_i u_i \quad \text{in } \mathbb{R}^{n-1},\ t > 0,\ i = 1, 2$$

- Parabolic in sense of Petrovskii if $\sigma + u_1 \partial_1 \sigma + u_2 \partial_2 \sigma \geq 0$
- Fulfilled if e.g. $\sigma(u)^2 = 2a(u_1/u_2)$ for some function a

Partial averaging in economics

$$\partial_t u_i = \Delta\big(a(u_1/u_2)u_i\big) + \mu_i u_i \quad \text{in } \mathbb{T}^d,\ t>0, \quad u_i(0)=u_i^0$$

or $\partial_t u = \operatorname{div}(A(u)\nabla u)$

- Assumptions: $a \in C^1(\mathbb{R})$, $a(r) \ge r|a'(r)|$, $a(r) \ge a_0/(r^p + r^{-p})$, examples: $a(r) = r^p$ for $0 < p \le 1$, $a(r) = 1/r$
- Nonstandard entropy: $\alpha \ge p+4$

$$H(u) = \int_{\mathbb{T}^d} h(u)dx,\ h(u) = \left(\frac{u_1}{u_2}\right)^\alpha u_1^2 + \left(\frac{u_2}{u_1}\right)^\alpha u_2^2 + \sum_{i=1}^2 (u_i - \log u_i)$$

- Entropy production:

$$\frac{dH}{dt} + \int_{\mathbb{T}^d}\left(\left(\frac{u_1}{u_2}\right)^{\alpha-p} + \left(\frac{u_2}{u_1}\right)^{\alpha-p}\right)(|\nabla u_1|^2 + |\nabla u_2|^2)dx \le C(\mu_1,\mu_2)H$$

- Properties: h convex, $h''(u)A(u)$ positive definite
- Yields global existence of weak solutions (A.J.-Zamponi 2016)

The story just began...

Topics in progress:

- General reaction terms: global existence of renormalized solutions (X. Chen-A.J. 2017)
- Weak-strong uniqueness of renormalized solutions (X. Chen-A.J. 2018)
- Structure-preserving numerical schemes (A.J.-Schuchnigg 2017, Chainais-Cancés-Gerstenmayer-A.J. 2018)
- Derivation from many-particle Markov processes (Fontbona-Méléard 2015, Moussa 2017, Daus-Desvillettes-Dietert 2018)
- Coupling with fluid models/thermodynamics (Druet et al. 2017)

Open questions:

- Do global weak solutions to n-species population model exist without detailed balance, for all $a_{ij} > 0$?
- How large is class of cross-diffusion systems with entropy structure?
- Is there any regularity theory beyond duality methods?