Spectral inequality for the Schrödinger equation $-\triangle_g + V(x)$ in R^d

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Marrakech, April 19, 2018
Premier (?) Congrès Franco Marocain
de Mathématiques Appliquées

Outline

1. Spectral inequality on a compact manifold
2. Spectral inequality for $-\triangle$ in R^d, a short review
3. Main result
4. Sketch of proof
   - Interpolation for holomorphic functions
   - Holomorphic extensions

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Spectral inequality on a compact manifold

Let (M, g) be a smooth connected compact manifold. Let $\triangle_g$ be the (negative) Laplace operator on M, and let

$$-\triangle_g e_j = \lambda_j e_j, \quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$$

be its spectral decomposition. The following spectral inequality was proved by Jerison-Lebeau and Lebeau-Zuazua in 1998-1999.

Theorem

Let $\omega \subset M$ be a non void open subset of M. There exists constants $A = A(\omega) > 0, C = C(\omega) > 0$ such that for all $\lambda > 0$ and all sequence $\{z_j\}_{j \in \mathbb{N}}$ of complex numbers, one has

$$\sum_{\lambda_j < \lambda} |z_j|^2 \leq A e^{C\lambda^{1/2}} \int_\omega |\sum_{\lambda_j < \lambda} z_j e_j(x)|^2 d_g x \ . \tag{1.1}$$

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In the rest of the talk, we will work in $\mathbb{R}^d$, and $\omega \subset \mathbb{R}^d$ will denote a measurable set satisfying the density assumption:

$$\exists R, \delta > 0, \quad \text{such that} \quad \inf_{x \in \mathbb{R}^d} mes \{t \in \omega, \ |x - t| < R\} \geq \delta. \tag{2.1}$$

Theorem

Let $\omega \subset \mathbb{R}^d$ be a measurable set satisfying the geometric condition (2.1). There exists constants $A = C(\omega), C = C(\omega) > 0$ such that the following hods true.

For all $\mu > 0$ and all $f \in L^2(\mathbb{R}^d)$, such that

$$support\ (\hat{f}) \subset \{\xi \in \mathbb{R}^d, |\xi| \leq \mu\},$$

where $\hat{f}$ is the Fourier transform of f, one has

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq A e^{C\mu} \int_\omega |f(x)|^2 dx. \tag{2.2}$$

Using techniques from Harmonic Analysis, Logvinenko and Sereda proved (*Tero. Funk. Anal. i Prilozen*, 1974) that in 1-d, the condition

$$E \subset \mathbb{R} \text{ measurable s.t. } \exists \gamma > 0,\ a > 0 \text{ s.t. } \frac{mes(E \cap I)}{mes(I)} \geq \gamma \qquad (2.3)$$

whenever I is an interval of length a, is sufficient to ensure that, when support$(\hat{f}) \subset [-b, b]$:

$$\exists C = C(\gamma, a, b) > 0 \text{ s.t. } \int_E |f(x)|^2 dx \geq C \|f\|^2_{L^2(\mathbb{R})}. \qquad (2.4)$$

On the other hand, the authors were not able to quantify the dependence of C with respect to the parameters a, b, γ. This was achieved in the one-dimensional case by Kovrojkine (*The Uncertainty principle for relatively dense sets and lacunary spectra*, 2002) where the author proves that

$$\exists K > \gamma \text{ such that } C(\gamma, a, b) = \left(\frac{\gamma}{K}\right)^{ab+1},$$

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Let g be a given Riemannian metric on $\mathbb{R}^d$. Let Δ_g be the (negative) Laplace operator defined by the metric g. Let $V = V(x)$ a real potential function such that $\lim_{x\to\infty} V(x) = 0$.

One defines the Schrödinger operator associated to (g, V) by

$$H_{g,V} := -\frac{1}{2}\Delta_g + V(x) \tag{3.1}$$

With reasonable hypothesis on (g, V), $H_{g,V}$ is a (bounded from below) unbounded self adjoint operator in $L^2(\mathbb{R}^d)$, and its spectrum $\sigma(g, V) \subset [E_0, \infty[$ satisfies the following.

- $\sigma(g, V) \cap]-\infty, 0[$ is purely discrete with eigenvalues of finite multiplicity, and its only possible accumulation point is 0.
- $\sigma(g, V) \cap]0, \infty[$ is absolutely continuous.

For $E \in \mathbb{R}$, we denote by Π_E the spectral projector on $]-\infty, E[$ associated to $H_{g,V}$. .

We will assume that (g, V) satisfy the following hypothesis (**H**)

- The metric g and the potential V are real analytic and there exists $a > 0$ such that they extend holomorphicaly in the complex domain $U_a = \{|Im(z)| < a\}$.
- One has $g = Id + \tilde{g}$, where $\tilde{g}$ is a symbol of degree < 0 in U_a.
- V is a symbol of degree < 0 in U_a.

– Observe that even in the case $g = Id$, the assumption on the potential V allows long range perturbation. Short range perturbations are associated to potentials V which are symbols of degree < -1 in U_a. For the analysis of scattering theory for long range perturbation, we refer to Hörmander, in *The analysis of linear pde's* vol 4, ch. XXX.
– Observe also that the metric g may have trapped trajectories.

Main result

For $E \in \mathbb{R}$, we define $E_{\pm}^{1/2}$ by

$$E_{\pm}^{1/2} = \sqrt{E} \quad \text{for} \quad E \geq 0, \qquad E_{\pm}^{1/2} = \pm i\sqrt{|E|} \quad \text{for} \quad E < 0.$$

Theorem

Let (g, V) satisfying hypothesis $(\mathbf{H})$. There exists constants $A = A(\omega, g, V), C = C(\omega, g, V) > 0$ such that for all $E \in \mathbb{R}$ and for all $f \in L^2(\mathbb{R}^d)$, one has

$$\|\Pi_E f\|_{L^2(\mathbb{R}^d)} \leq A|e^{CE_{\pm}^{1/2}}| \; \|\Pi_E f\|_{L^2(\omega)}. \qquad (3.2)$$

Main result

Observe that under the hypothesis (**H**), which allows long range perturbation, we may have

$$dim(range(\Pi_0)) = \infty.$$

In particular, inequality 3.2 implies that any function $f \in range(\Pi_0)$ satisfies:

$$f(x) = 0 \text{ for all } x \in \omega \Rightarrow f = 0.$$

In fact, in the course of the proof, we will show that any $f \in range(\Pi_0)$ extends holomorphicaly in U_a for $a > 0$ small enough. Thus uniqueness holds true for **any** measurable set ω of positive measure.

Outline

The first main ingredient is to use the following " classical" interpolation inequality for holomorphic functions. Let $\omega \subset \mathbb{R}^d$ satisfying the hypothesis (**H**). There exists constants $C_{int} = C_{int}(\omega, a) > 0$ and $\delta = \delta(\omega, a) \in (0,1)$ such that

$$\int_{\mathbb{R}^d} |f|^2 dx \leq C_{int} \left(\int_\omega |f|^2 dx \right)^\delta \left(\int_{U_a} |f|^2 |dz| \right)^{1-\delta}, \tag{4.1}$$

for any $f \in L^2(U_a) \cap \mathcal{H}(U_a)$ satisfying

$$\sup_{0 \leq b < a} \int_{|y|=b} |f(x+iy)|^2 dx < \infty \tag{4.2}$$

Poisson kernel

We denote by $d\pi_E(x,y)$ the kernel of the spectral measure of $H_{g,V}$, i.e

$$d\pi_E(f,g) = \int_{\mathbb{R}^d\times\mathbb{R}^d} f(x)\overline{g}(y)d\pi_E(x,y), \quad f,g \in L^2(\mathbb{R}^d).$$

Recall that $d\pi_E(f,f)$ is the positive measure on the line $E \in \mathbb{R}$ equal to the derivative of the left continuous and non decreasing function

$$E \mapsto \|\Pi_E(f)\|^2_{L^2}.$$

The Poisson kernel $\mathbb{P}_{s,\pm}(x,y)$ is the smooth function on $]0,\infty[\times\mathbb{R}^d\times\mathbb{R}^d$ given by the formula

$$\mathbb{P}_{s,\pm}(x,y) = \int_{\mathbb{R}} e^{-sE_\pm^{1/2}} d\pi_E(x,y). \tag{4.3}$$

For any $f \in L^2(\mathbb{R}^d)$, the smooth function on $]0,\infty[\times\mathbb{R}^d$ defined by $u(s,x) = \int_{\mathbb{R}^d} \mathbb{P}_{s,\pm}(x,y)f(y)dy$ satisfies the elliptic boundary problem

$$(-\partial_s^2 + H_{g,V})u = 0, \quad \lim_{s\to 0^+} u(s,x) = f(x) \text{ in } L^2(\mathbb{R}^d). \tag{4.4}$$

An analog of the Boutet de Monvel theorem

The second main ingredient is the following Lemma which is an extension of the analytic form of the famous result of Louis Boutet de Monvel on the extension in the complex domain of eigenfunctions of an elliptic operator on a compact riemannian manifold.

Lemma

Let $u(s, x)$ be a solution of the elliptic equation $(-\partial_s^2 + H_{g,V})u = 0$ on $]0, \infty[\times\mathbb{R}^d$. Then for $a > 0$ and $\delta > 0$ small enough, u extends holomorphically in the open set

$$\mathcal{B}_a = \{(s, z) \in \mathbb{C} \times \mathbb{C}^d \ Re(s) > 0, |Im(z)| \leq min(a, \delta|Re(s)|)\},$$

Zerner Lemma

The proof of the above Lemma uses the following classical Zerner Lemma.

Lemma

(Zerner) *Let* $Q(z, \partial_z) = \sum_{\alpha, |\alpha| \leq m} q_\alpha(z) \partial_z^\alpha$ *be a linear differential operator with holomorphic coefficients defined near* 0 *in* $\mathbb{C}^N$ *and let* $q(z, \zeta) = \sum_{|\alpha|=m} q_\alpha(z)\zeta^\alpha$ *be its principal symbol. Let* $f : \mathbb{C}^N \to \mathbb{R}$ *be a* C^1 *function such that* $f(0) = 0$ *and such that, with* $\zeta_0 = 2i\partial f(0)$, *one has* $q(0, \zeta_0) \neq 0$. *Then, if* $u(z)$ *is an holomorphic function defined in a half-neighborhood of* 0 *in* $f < 0$, *such that* $Q(u)$ *extends holomorphically near* 0, *then* u *extends holomorphically near* 0.

Recall that the Zerner Lemma was the starting block for the introduction by M. Kashiwara of micro-hyperbolic tools in the analysis of PDE's, and it leads further to the construction by M. Kashiwara and P. Schapira of the so called *Microlocal Sheaf Theory*.

Simple pseudo-differential calculus

In order to prove our main result, we will also use the following simple " pseudo-differential calculus" defined by the symbolic calculus of the self adjoint unbounded operator $H_{g,V}$. For a measurable and bounded function $\chi(E)$ on $\mathbb{R}$, the operator $A(\chi) = \chi(H_{g,V})$ has the following kernel:

$$A(\chi)(x,y) = \int_{\mathbb{R}} \chi(E) d\pi_E(x,y).$$

In particular, one has

$$\mathbb{P}_{s,\pm}(x,y) = A(e^{-sE_{\pm}^{1/2}}).$$

Then, we use the following elementary fact: For any given λ, and any f such that $\Pi_\lambda(f) = f$, one has

$$\mathbb{P}_{s,\pm} f = \mathbb{P}_{s-\delta,\pm} A(\chi)(f), \quad \chi(E) = e^{\delta E_{\pm}^{1/2}} 1_{E<\lambda}$$

and we use Lemma 4.1.

Frères humains, qui après nous vivez,
N'ayez les coeurs contre nous endurcis,
Car, si pitié de nous pauvres avez,
Dieu en aura plus tôt de vous mercis.

François Villon
La ballade des pendus, 1462.

Hier Encore

Puisqu'en France, tout finit par des chansons:

Hélas, mille fois Hélas ....