STABILITY PROBLEMS IN KINETIC THEORY FOR SELF-GRAVITATING SYSTEMS

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Premier Congrès Franco-Marocain de Mathématiques Appliquées

Marrakech 16-20 Avril 2018

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Outline

The N-body problem

- Newton's equations for N interacting bodies

$$\dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = -\sum_{j\neq i} \nabla V(x_i(t) - x_j(t)).$$

- Newton or Coulomb potential

$$V(r) = \pm\frac{1}{r}.$$

- For $N >> 10^6$: Fluid dynamics description.
- For N large but not too much ($N \sim 10^6$), a statistical description is more appropriate. For galaxies, a collisionless kinetic description is the most popular in astrophysics.
  Distribution function of bodies: $f(t, x, v)$. Stellar dynamics started to be developed at the beginning of XX centuary.

The classical Vlasov-Poisson equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0, \qquad f(t=0,x,v) = f_0(x,v)$$

$$\phi_f(t,x) = \frac{\gamma}{4\pi} \int_{\mathbb{R}^3} \frac{\rho_f(t,y)}{|x-y|} dy, \qquad \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,v) dv.$$

Poisson equation: $\Delta \phi_f = \gamma \rho_f.$

- Gravitational systems, $\gamma = +1$: galaxies, star clusters, etc.
- Systems of particles , $\gamma = -1$: charged particles with Coulomb interactions.
- Some extensions
  - Relativistic VP: replace v by $\frac{v}{\sqrt{1+|v|^2}}$:
  - Vlasov-Manev (1920): replace the interaction potential $\frac{1}{|x-y|}$ by $\frac{1}{|x-y|} + \frac{1}{|x-y|^2}$. Manev, 1920.
  - Vlasov-Einstein: Couple Vlasov with relativistic metrics, Einstein equations.

Basic properties

- Conservation of the energy: $\mathcal{H}(f) = E_{kin}(f) - \gamma E_{pot}(f)$

$$E_{kin}(f) = \frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f dx dv, \qquad E_{pot}(f) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_x \phi_f|^2 dx$$

- Conservation of the Casimir functionals $\displaystyle\int_{\mathbb{R}^6} G(f) dx dv$.
- Galilean invariance: f solution $\Longrightarrow$ $f(t, x + v_0 t, v + v_0)$ is also a solution.
- Scaling symmetry: f solution $\Longrightarrow$ $\frac{\mu}{\lambda^2} f\left(\frac{t}{\lambda\mu}, \frac{x}{\lambda}, \mu v\right)$ solution too.
- In the case of spherically symmetric solutions $f(t, |x|, |v|, x \cdot v)$, the angular momentum $\int_{\mathbb{R}^6} |x \times v|^2 f dx dv$ is also conserved.

Cauchy Theory in the gravitational case

A key interpolation inequality:

$$E_{pot}(f) \leq CE_{kin}(f)^a \left(\int f\right)^b \left(\int f^p\right)^c \quad \text{for} \quad p \geq p_{crit}$$

Existence of solution as long as the kinetic energy is controlled.

- Classical VP: $a = 1/2$. Global existence: Arsen'ev 1975, Illner-Neunzert 1979, Horst-Hunze 1984, Diperna-Lions 1988, Pfaffelmoser 1989, Lions-Perthame 1991, Schaeffer 1991, Loeper (2006), Pallard 2012, ...
- Relativistic VP: $a = 1$. Blow-up in finite time is possible: Glassey-Schaeffer 1986.
- Vlasov-Manev: $a = 1$. Blow-up in finite time is possible: Bobylev-Dukes-Illner-Victory 1997.

A class of steady states

$$v \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0.$$

In the plasma case ($\gamma = -1$) the only solution is 0. In the gravitational case, the general resolution is an open question.

- Isotropic galactic models:

$$f(x,v) = F\left(\frac{|v|^2}{2} + \phi_f(x)\right), \quad \Delta\phi_f(x) = \int_{\mathbb{R}^3} F\left(\frac{|v|^2}{2} + \phi_f(x)\right) dv.$$

- Anisotropic models:

$$f(x,v) = F\left(\frac{|v|^2}{2} + \phi_f(x), |x \times v|^2\right).$$

  If spherical symmetry $f := f(|x|, |v|, x \cdot v)$, then the Jeans theorem ensures that all spherically symmetric steady states are of this form (Batt-Faltenbacher-Horst 86).

- Two important examples are:
  - Polytropes: $F(e) = C(e_0 - e)_+^p$.
  - The King model: $F(e) = \alpha\left(\exp(\beta(e_0 - e)) - 1\right)_+$.

What Stability means?

- The energy space:

$$\mathcal{E}_j = \{f \text{ such that } \|f\|_{\mathcal{E}_j} = \int \left(1 + |v|^2\right) f dx dv + \int j(f) dx dv < \infty\}.$$

- A steady state f_0 is said to be stable through the VP flow if for all $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\|f(0) - f_0\|_{\mathcal{E}_j} < \eta \implies \forall t \geq 0, \ \|f(t) - f_0\|_{\mathcal{E}_j} < \varepsilon.$$

$f(t)$ being the solution to VP associated with the initial data f_0.

- Galilean invariance: orbital stability

$$\forall t \geq 0, \ \ \exists x_0(t) \in \mathbb{R}^3, \ \ \ \|f(t, \cdot + x_0(t), \cdot) - f_0\|_{\mathcal{E}_j} < \varepsilon.$$

Physics literature: Antonov, Lynden-Bell (1960'), Doremus-Baumann-Feix (1970'), Kandrup-Signet (1980'), Aly-Perez (1990'), ..., see Binney-Tremaine.

Mathematics literature: Two last decades: Wolansky, Guo, Rein, Dolbeault, Lin, Hadzic, Sanchez, Soler, L-Méhats-Raphaël, Rigault, Fontaine ...

Homogeneous steady states

A different important context: If periodic domain in space: Homogeneous steady states:

$$f(x,v) = g_0(|v|).$$

Asymptotic stability under Penrose conditions: Landau damping, Mouhot-Villani.

Linear Stability

$$f_0(x,v) = F\left(\frac{|v|^2}{2} + \phi_0(x)\right)$$

- Linearized VP around f_0: $f = f_0 + g$

$$\partial_t g + v \cdot \nabla_x g - \nabla_x \phi_0 \cdot \nabla_v g = \nabla_x \phi_g \cdot \nabla_v f_0$$

- The linearized Casimir functional are preserved

$$\int \chi\left(\frac{|v|^2}{2} + \phi_0(x)\right) g dx dv$$

- The linearized Hamiltonian is preserved

$$\mathcal{H}(g) = \int \left(\frac{|v|^2}{2} + \phi_0\right) g dx dv.$$

Linear stability

- The Energy-Casimir functional:

$$\mathcal{H}_j(f) = \int \frac{|v|^2}{2} f - \frac{1}{2} \int |\nabla_x \phi_f|^2 dx + \int j(f) dxdv$$

- Second derivative around f_0 with $j' \circ F = -Id$:

$$\mathcal{A}(g,g) = \frac{1}{2} \int j''(f_0) g^2 dxdv - \frac{1}{2} \int |\nabla_x \phi_g|^2 dx.$$

- This is preserved by the linearized VP flow.
- Does $\mathcal{A}(g,g)$ control some strong norms of the perturbation g ?
- Degeneracy due to translation invariance:

$$\mathcal{A}\left(\partial_{x_i} f_0, \partial_{x_i} f_0\right) = 0, \quad i = 1, 2, 3.$$

Linear stability - Antonov inequality

- Aspherical perturbations, Second Antonov law (1960'). Any isotropic equilibrium $f_0 = F\left(\frac{|v|^2}{2} + \phi_0(x)\right)$ with $F' < 0$ is stable under *aspherical* perturbations and up to space translation schifts:

$$\mathcal{A}(g,g) > 0, \quad \text{for all aspherical } g \text{ with } \int g \partial_{x_i} f_0 \, dxdv = 0.$$

- DOREMUS-FEIX-BAUMANN (1971): Any equilibrium $f_0 = F\left(\frac{|v|^2}{2} + \phi_0(x)\right)$ with $F' < 0$ is stable under spherical perturbation. The main tool: the so called Antonov's inequality

$$\mathcal{A}(g,g) \geq \int_{supp(f_0)} \frac{\xi^2}{|F'|} \frac{\phi_0'(r)}{r} dxdv,$$

for all spherically symmetric g such that

$$\int \chi\left(\frac{|v|^2}{2} + \phi_0(x), |x \times v|^2\right) g = 0, \quad \forall \chi,$$

or equivalently $\quad g = v \cdot \nabla_x \xi - \nabla_x \phi_0 \cdot \nabla_v \xi, \quad$ for some ξ.

Statements of the linear stability results

- GENERAL PERTURBATIONS All isotropic steady states

$$f_0(x,v) = F\left(\frac{|v|^2}{2} + \phi_0(x)\right)$$

which are decreasing functions of the microscopic energy are stable under general perturbations, up to space translation shifts.

- SPHERICAL PERTURBATIONS All anisotropic steady states

$$f_0(x,v) = F\left(\frac{|v|^2}{2} + \phi_0(x), |x \times v|^2\right)$$

which are decreasing functions of the microscopic energy are stable under spherical perturbations.

- Optimal: Non spherical perturbations of anisotropic steady states may give instabilities, Binney-Tremaine.

Outline

A general strategy in a variational approach

A subclass of steady states: minimizers of some functional preserved by the flow under constraints also preserved by the flow.

- Consider a variational problem of the form:

$$\inf_{|f|_{L^1}=M,\dots} \mathcal{H}(f)+\int j(f), \qquad j \text{ convex.}$$

- Existence of the infimum and of the minimizers: interpolation inequalities + compactness of a particular minimizing sequence.
- Minimizers (denoted by f_0) are steady states: Euler-Lagrange equations are

$$\frac{|v|^2}{2}+\phi_0(r)+j'(f_0)=\lambda \quad \text{on the support of } f_0$$

$$\Delta\phi_0=\int_{\mathbb{R}^3}(j')^{-1}\left(\lambda-|v|^2/2-\phi_0(x)\right)_+ dv.$$

- Radial symmetry of the minimizers. Two ways: Gidas-Ni-Nirenberg theorem or case of equality in Riesz rearrangement inequalities.

Link between stability and variational approaches

Compactness of all minimizing sequences $\implies$ Stability of the set of minimizers.

Minimizing sequence: $\int f_n \to M_1$ and $\mathcal{H}(f_n) + \int j(f_n) \ \to \ I(M_1)$ as $n \to \infty$.

Scheme of a proof (contradiction argument):

- Let $f^n(0) \to f_0$ in the energy space, with $\|f^n(t_n) - f_0\| > \varepsilon$, for some t_n.
- Conservation of the Hamiltonian and of the constraints, $\forall t$:

$$\mathcal{H}(f^n(t)) = \mathcal{H}(f^n(0)) \to \mathcal{H}(f_0)$$

  $f^n(t_n)$ is a minimizing sequence and then *strong compactness* implies

$$f^n(t_n) \to \text{a minimizer}$$

- If one has *uniqueness* of the minimizer then a contradiction.
- *Natural instabilities:* Galilean invariance. Initial data $f_0(x, v + v_0)$ leads to $f_0(x + v_0 t, v + v_0)$ $\implies$ *Orbital stability* is *compactness* up to *translation shifts* only.

The one constraint approach

Minimize the Energy-Casimir functional under the constraint of a given mass.

$$\inf_{|f|_{L^1}=M}\left[\mathcal{H}(f)+\int j(f)\right]=I(M).$$

- Existence of the infimum: use the interpolation inequality (valid for $p>9/7$) and $j(t)\geq Ct^p$

$$\mathcal{H}(f)\geq E_{kin}-C\left(\int j(f)\right)^{1/(3p-3)}E_{kin}^{1/2}+\int j(f).$$

$$\mathcal{H}(f)\geq -\frac{C}{4}\left(\int j(f)\right)^{2/(3p-3)}+\int j(f),$$

  which is bounded from below if and only if $p>5/3$.
- The original range of p (which is $p>9/7$), can be recovered as follows: replace $\int j(f)$ by $\left(\int j(f)\right)^{7/3}$. See also Guo-Rein (other variational pb).

The one constraint approach – Compactness

- As a first step, let us prove the compactness of spherically symmetric minimizing sequences. f_n:

$$\|f_n\|_{L^1} \to M, \qquad \mathcal{H}(f_n) + \int j(f_n) \to I(M).$$

  - Weak compactness in L^p, $p > 5/3$: f_n converges weakly to f in L^p.
  - Spherical symmetry $\implies$ strong convergence of the potential energy (local compactness + explicit decay of the potential energy)
  - $I(M)$ is a strictly decreasing function of M, by scaling arguments.
  - $I(\|f\|_{L^1}) \leq \mathcal{H}(f) + \int j(f) \leq I(M)$ by lower semi-continuity.
  - Saturation of the constraints, and strong convergence of f_n in the energy space to f which is a minimizer.
- The general case when f_n is not spherically symmetric, is based on the well-known *concentration-compactness lemma, Lions 1984*: one gets *Compactness up to translations*. Scaling arguments are important in the analysis!

Stability and uniqueness of the minimizer

- We then get orbital stability of the set of minimizers. Examples are *polytropes or generalized polytropes* but NOT *the King model* .
- One could think that the uniqueness or the isolatedness of the minimizers is necessary. In fact it is not! Note that the uniqueness of the minimizers fails in general.
- Use the rigidity of the flow.

A uniqueness lemma - The rigidity of the VP flow

Uniqueness Lemma (L- Méhats and Rigault, 2012)

Consider two distribution functions of the form

$$f_1(x,v) = F\left(\frac{|v|^2}{2} + \psi_1(x)\right), \qquad f_2(x,v) = F\left(\frac{|v|^2}{2} + \psi_2(x)\right),$$

where the common profile F is strictly decreasing and the potentials are spherically symmetric and nondecreasing.

If f_1 and f_2 are equimeasurable then $f_1 = f_2$

This implies that two equimeasurable minimizers are equal.

- This is quite general result because it does not use the Euler Lagrange equation satisfied by the minimizers, but rather the rigidity of the flow (equimeasurability).
- It can be applied to relativistic contexts, with Poisson or Manev potentials, and with arbitrary (but finite) number of constraints.

The insufficiency of variational approaches

Consider the set of all spherically symmetric solutions to

$$\Delta\psi_\alpha = \int j'^{-1}\left(-\frac{|v|^2}{2} - \psi_\alpha(x)\right) dv, \qquad \psi_\alpha(0) = \alpha, \alpha < 0.$$

Then the corresponding potential of a steady state is

$$\phi_\alpha(x) = \psi_\alpha(x) - \psi_\alpha(+\infty).$$

We denote the corresponding steady state by f_α.

- Any minimizer is an element of this family: take a mass $M > 0$, the corresponding minimizer f of the one constraint problem is of the form

$$f_\alpha = j'^{-1}\left(\lambda - \frac{|v|^2}{2} - \phi(x)\right).$$

  Then set $\alpha = \phi(0) - \lambda$: we have $\psi_\alpha(x) = \phi(x) - \lambda$ and $-\lambda = \psi_\alpha(+\infty)$.
- However, not all the steady states f_α are minimizers.

The Lieb-Yau variational principle

- The Lieb-Yau principle (1987): the mass $M(\alpha)$ is decreasing in α along the minimizers
- Consequence 1: If $\alpha \mapsto M(\alpha)$ is decreasing then all the f_α are minimizers.
- Consequence 2: If $\alpha \mapsto M(\alpha)$ is not decreasing then all the f_α are not minimizers.

Remark: For polytropic profiles $j(f) = f^p$, it is easy to show that $M(\alpha)$ is decreasing, so all steady states are minimizers.

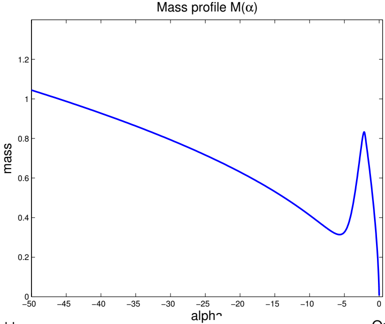
Numerical counterexample

Consider the function $j(f)$ from [Schaeffer 2004]:

$$j'(f) = \begin{cases} c_1\, f^4 & \text{if} \quad 0 \leq f \leq 0.25 \\ c_2\, f^{0.01} & \text{if} \quad 0.25 \leq f \leq 4 \\ c_3\, f^2 & \text{if} \quad 4 \leq f \end{cases}$$

Then from numerical simulations, one observes that:

- The function $M(\alpha)$ is not decreasing.
- The one constraint problem does not cover all steady states and displays non uniqueness for some mass M_1^*.


Mass profile M(α)
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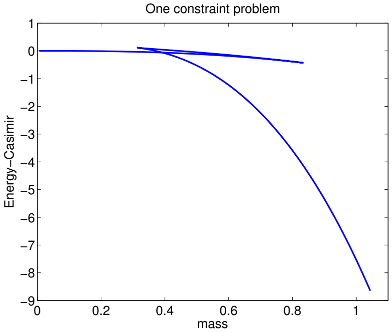
One constraint problem
Energy−Casimir
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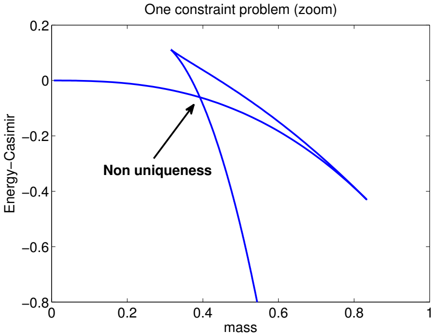
One constraint problem (zoom)
Energy−Casimir
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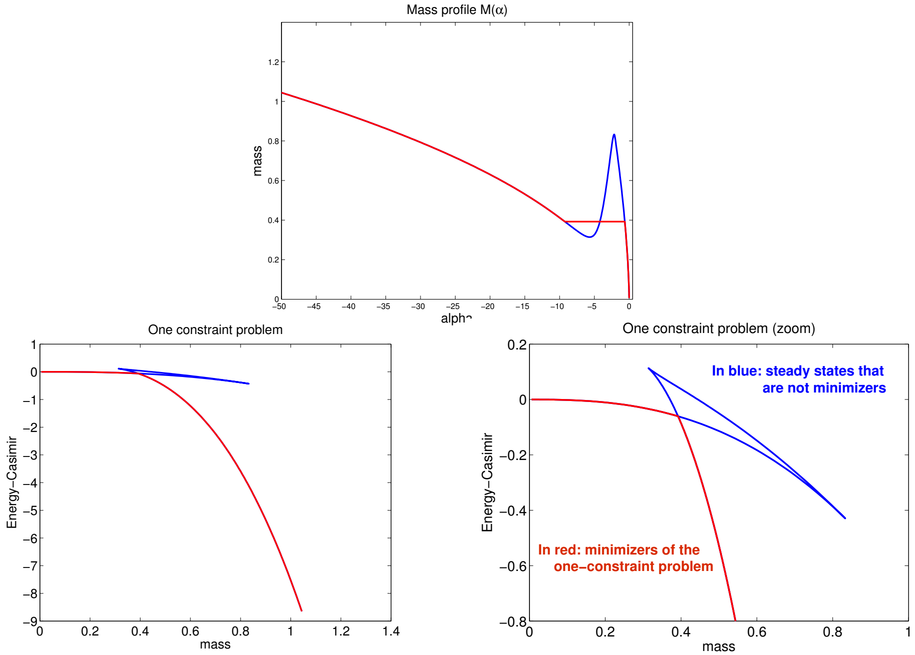
Mass profile M(α)
mass
alpha
One constraint problem
Energy−Casimir
mass
One constraint problem (zoom)
In blue: steady states that are not minimizers
In red: minimizers of the one−constraint problem

The two constraints problem [1]

$$\inf_{\|f\|_{L^1} = M,\ \|j(f)\|_{L^1} = M_j} \mathcal{H}(f) = I(M, M_j).$$

- The two-constraints problem provides stability of a two-parameters class of minimizers which, for all j, contains the set provided by one constraint problem.
- In fact, there are some Casimir functions j for which, these two sets are the same: polytropes.

  But there are some for which the one constraint set is strictly included in the two-constraints set. The difference between the two sets may be an open set of steady states.
- The two-constraint problem is still not sufficient to recover all the decreasing steady states because
  - of the assumptions $j(t) \geq t^p, p > 9/7$, and
  - it can be shown numerically that it does not cover all the steady states with a given profile.

[1] L- Méhats-Raphaël, 2008, 2009

Outline

Statement of the stability result

(i) $f_0(x,v) = F\left(\frac{|v|^2}{2} + \phi_0(x)\right)$ is $\mathcal{C}^0$ and compactly supported.

(ii) F is C^1 on $]-\infty, e_0[$ with $F' < 0$ and, on $[e_0, +\infty[$, $F(e) = 0$.

Theorem (L, Méhats, Raphaël. 2012)

Orbital stability of f_0. *For all $\varepsilon > 0$, for all $M > 0$, there exists $\eta > 0$ such that the following holds true. Let $f_{in} \in L^1 \cap L^\infty$, with $f_{in} \geq 0$ and $|v|^2 f_{in} \in L^1$, be such that*

$$\|f_{in} - f_0\|_{L^1} < \eta, \quad \mathcal{H}(f_{in}) \leq \mathcal{H}(f_0) + \eta \quad \|f_{in}\|_{L^\infty} < \|f_0\|_{L^\infty} + M,$$

then there exists a translation shift $z(t)$ such that the corresponding weak solution $f(t)$ to VP satisfies: $\forall t \geq 0$,

$$\|(1 + |v|^2)(f(t,x,v) - f_0(x - z(t), v))\|_{L^1(\mathbb{R}^6)} < \varepsilon.$$

A first idea would be to introduce a variational problem with an infinite number of constraints. Not sure that this covers all the steady sates considered here. Rather try to control directly the distribution function by using Hamiltonian and all the Casimirs.

Equimeasurability and Schwarz rearrangement

- **Equimeasurability:** consider the set $\mathrm{Eq}(f_0)$ of nonnegative functions $f \in L^1 \cap L^\infty$ that are equimeasurable with f_0:

$$\int G(f(x,v))dxdv = \int G(f_0(x,v))dxdv, \qquad \forall G$$

or

$$\mu_f(\lambda) = \text{meas}\{f(x,v) > \lambda\} = \text{meas}\{f_0(x,v) > \lambda\} = \mu_{f_0}(\lambda), \qquad \forall \lambda \geq 0.$$

- **The standard Schwarz symmetrization.** Let $f \in L^1(\mathbb{R}^d)$, then there exists a unique nonincreasing function $f^* \in L^1(\mathbb{R}^d)$ of $|x|$, such that f^* is equimeasurable with f:

$$f^*(x) = f^\sharp\left(|B_d(0,|x|)|\right), \qquad f^\sharp \text{ is the pseudo inverse of } \mu_f.$$

- if f is a solution of the Vlasov system then:

$$f(t)^* = f(0)^*.$$

Two main steps in the original proof

- Reduce the Hamiltonian to a functional of ϕ only:

$$\mathcal{H}(f) - \mathcal{H}(f_0) \geq \mathcal{J}(\phi_f) - \mathcal{J}(\phi_0) - C\|f^* - f_0^*\|_{L^1}.$$

 and get Local quantitative control of the potential:

$$\inf_{z\in\mathbb{R}^3} \|\nabla\phi_f - \nabla\phi_0(\cdot - z)\|_{L^2}^2 \leq C\left[\mathcal{H}(f) - \mathcal{H}(f_0) + \|f^* - f_0^*\|_{L^1}\right]$$

 For all $f \in \mathcal{E}$ such that ϕ_f is in a neighborhood U of ϕ_0.

- Local compactness of the full distribution function:
 Let f_n be any sequence in the energy space such that ϕ_{f_n} is in U.
 Assume that

$$f_n^* \to f^* \text{ in } L^1, \qquad \mathcal{H}(f_n) \to \mathcal{H}(f_0).$$

 Then there exists a sequence $z_n \in \mathbb{R}^3$ such that

$$\|(1 + |v|^2)(f_n(x,v) - f_0(x - z_n, v)\|_{L^1(\mathbb{R}^6)} \to 0.$$

Rearrangement with respect to the microscopic energy.

Let $\phi(x)$ be a potential field.
Let $f \in L^1 \cap L^\infty(\mathbb{R}^6)$, then we may define its rearrangement with respect to

$$e(x, v) = \frac{|v|^2}{2} + \phi(x).$$

which we denote $f^{*\phi}$. It is

- a nonincreasing function of $\frac{|v|^2}{2} + \phi(x)$;
- such that $f^{*\phi} \in \mathrm{Eq}(f)$.

Caracterisation: Our steady states are fixed points of this transformation

$$f_0^{*\phi_0} = f_0$$

Rearrangement with respect to the microscopic energy.

EXPLICIT CONSTRUCTION OF $f^{*\phi}$

$$f^{*\phi}(x,v) := f^\sharp\left(a_\phi\left(\frac{|v|^2}{2}+\phi(x)\right)\right)\mathbb{1}_{\frac{|v|^2}{2}+\phi(x)<0}$$

where a_ϕ is the Jacobian function defined by

$$\begin{aligned} a_\phi(e) &= \text{meas}\left\{(x,v)\in\mathbb{R}^6 : \frac{|v|^2}{2}+\phi(x)<e\right\} \\ &= \frac{8\pi\sqrt{2}}{3}\int_0^{+\infty}(e-\phi(x))_+^{3/2}\,dx \end{aligned}$$

The key monotonicity property

Lemma. *Let f be a distribution function and ϕ_f its Poisson potential. Then*

$$\mathcal{H}(f) \geq \mathcal{H}(f^{*\phi_f}).$$

Proof.
Denote $\widehat{f} = f^{*\phi_f}$. We have the decomposition

$$\mathcal{H}(f) = \mathcal{H}(\widehat{f}) + \frac{1}{2}\|\nabla\phi_f - \nabla\phi_{\widehat{f}}\|_{L^2}^2 + \int\left(\frac{|v|^2}{2} + \phi_f\right)(f - \widehat{f})dxdv.$$

By construction of $f^{*\phi_f}$, the green term is nonnegative. This is reminiscent from the following property of the standard Schwarz symmetrization:

$$\int_{\mathbb{R}^3} |x|f(x)dx \geq \int_{\mathbb{R}^3} |x|f^*(x)dx.$$

which is a consequence of the Hardy-Littlewood inequality: Hardy, Littlewood, Pólya: Inequalities, 1934. Lieb and Loss: Analysis.

$$\int f(x)g(x)dx \leq \int f^*(x)g^*(x)dx.$$

Reduction to a problem on the potential

$$\mathcal{H}(f) \geq -C\|f^* - f_0^*\| + \mathcal{J}(\phi_f) + \int \left(\frac{|v|^2}{2} + \phi_f\right)(f - f^{*\phi_f})dxdv.$$

$$\mathcal{J}(\phi) = \int \left(\frac{|v|^2}{2} + \phi(x)\right) f_0^{*\phi}(x,v)dxdv + \frac{1}{2}\|\nabla\phi\|_{L^2}^2$$

Two points:

- The red term $\mathcal{J}(\phi_f)$ only depends on the potential ϕ_f, and $\mathcal{J}(\phi_0) = \mathcal{H}(\phi_0)$. f^* is preserved by the flow.
- The green term is nonnegative and vanishes when $f = f_0^{*\phi_f}$.

$$\mathcal{H}(f) - \mathcal{H}(f_0) \geq \mathcal{J}(\phi_f) - \mathcal{J}(\phi_0) - \text{Invariants}.$$

Study of $\mathcal{J}$ and control of ϕ

$$\mathcal{J}(\phi) = \int \left(\frac{|v|^2}{2} + \phi(x) \right) f^{*\phi}(x, v) dx dv + \frac{1}{2} \|\nabla \phi\|_{L^2}^2$$

$$f^{*\phi}(x, v) = f_0^\sharp \left(a_\phi \left(\frac{|v|^2}{2} + \phi(x) \right) \right)$$

Proposition. *The quantity $\mathcal{J}(\phi) - \mathcal{J}(\phi_0)$ controls the distance of ϕ to the manifold of translated Poisson fields $\mathcal{M} = \{\phi_0(\cdot + z), \quad z \in \mathbb{R}^3\}$: in the vicinity of $\mathcal{M}$, we have*

$$\mathcal{J}(\phi) - \mathcal{J}(\phi_0) \geq C \inf_{z \in \mathbb{R}^3} \|\nabla \phi - \nabla \phi_0(\cdot - z)\|_{L^2}^2 \quad \textit{with } C > 0.$$

Proof. Based on a Taylor expansion. We differentiate twice the functional $\mathcal{J}$ with respect to ϕ and study the Hessian: it is nonnegative, and coercive on spherical functions.

Control of the whole distribution function by compactness

$$\mathcal{H}(f)-\mathcal{H}(f_0) \geq -C\|f^* - f_0^*\| + \mathcal{J}(\phi_f) - \mathcal{J}(\phi_0) + \int \left(\frac{|v|^2}{2} + \phi_f\right)(f - f^{*\phi_f})dxdv.$$

- Contro of the potential energy:

$$\mathcal{J}(\phi) - \mathcal{J}(\phi_0) \geq C \inf_{z\in\mathbb{R}^3} \|\nabla\phi - \nabla\phi_0(\cdot - z)\|^2_{L^2}.$$

- Compactness on the distribution function

If $\int \left(\frac{|v|^2}{2} + \phi_{f_n}\right)(f_n - f_n^{*\phi_n})dxdv \to 0$, and $f_n^* \to f_0^*$ in L^1 then

f_n strongly converges to f_0 in L^1.

- However: No quantitative information about the perturbation. Goal is to obtain a stability functional inequality of the generic form (up to symmetries of the system)

$$\|f - f_0\|^2_{L^1} \leq C\left(\mathcal{H}(f) - \mathcal{H}(f_0) + C\|f^* - f_0^*\|_{L^1}\right).$$

Generalized rearrangement

ML, 2016.

Let σ be a nonnegative measurable function of $\Omega \subset \mathbb{R}^d$, $d \geq 1$ such that for all $e \in [0, e_{max})$

$$\text{meas}\{x \in \Omega, \sigma(x) = e\} = 0.$$

Let

$$a_\sigma(e) = \text{meas}\{x \in \Omega, \sigma(x) < e\}, \quad a_\sigma(e_{max}) = |\Omega|.$$

For all $f \in L^1(\Omega)$, we define its rearrangement $f^{*\sigma}$ with respect to σ by

$$f^{*\sigma}(x) = f^\sharp(a_\sigma(\sigma(x)))\mathbb{1}_{\sigma(x)<e_{max}}, \quad \forall x \in \Omega,$$

In particular $f^{*\sigma}$ is the only decreasing function of $\sigma(x)$ which is equimeasurable with f.

Extended Hardy-Littlewood inequality

Let σ be as above. Then for any nonnegative functions $f, g \in L^1(\Omega)$ we have

$$\int_\Omega f(x)g(x)dx \leq \int_\Omega f^{*\sigma}(x)g^{*\sigma}(x)dx,$$

In particular

$$\int_\Omega \sigma(x)(f(x) - f^{*\sigma}(x))dx \geq 0.$$

Does this nonnegative quantity control some strong norm $\|f - f^{*\sigma}\|$?

- Weak answer: Saturating the inequality $\Longrightarrow$ Compactness

  if $\int_\Omega \sigma(x)(f_n(x) - f_n^{*\sigma}(x))dx \to 0,$ and if $\|f_n^{*\sigma} - f_0\|_{L^1} \to 0$ then

$$\|f_n - f_0\|_{L^1} \to 0.$$

- In the same spirit as in Burchard-Guo (JFA, 2004) concerning the Riez rearrangement inequality.

Refined HL inequalities

Refined HL inequality (ML-2016)

Let σ be as above and b_σ the pseudo inverse of a_σ. Then for any nonnegative function $f \in L^1(\Omega)$ we have

$$\|f - f^{*\sigma}\|_{L^1}^2 \leq K(f^*, \sigma) \int_\Omega \sigma(x)(f(x) - f^{*\sigma}(x))dx,$$

where $K(f^*, \sigma)$ is a constant depending only on f^* and σ. More generally, for any nonnegative $f, f_0 \in L^1(\Omega)$

$$\begin{aligned}(\|f - f_0^{*\sigma}\|_{L^1} + \|f_0\|_{L^1} - \|f\|_{L^1})^2 \quad &\leq K(f_0^*, \sigma)\left[\int_\Omega \sigma(x)(f(x) - f_0^{*\sigma}(x))dx\right. \\ &\left. + \int_\Omega \left(b_\sigma[2\mu_{f_0}(s)]\beta_{f^*, f_0^*}(s) - b_\sigma[\mu_{f_0}(s)]\beta_{f_0^*, f^*}(s)\right) ds\right],\end{aligned}$$

with $\beta_{f,g}(s) = \text{meas}\{x \in \Omega : f(x) \leq s < g(x)\}$.

A particular case:

Case of Schwarz symmetrization:

Corollary (L-2016)

For all $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $d \geq 1$, and all $0 \leq m \leq d$, we have

$$\int_{\mathbb{R}^d} |x|^m (f(x) - f^*(x)) dx \geq K_d \|f\|_{L^\infty}^{-m/d} \|f\|_{L^1}^{-1+m/d} \|f - f^*\|_{L^1}^2,$$

$$K_d = 2^{-1+m/d} \frac{m^2}{4d^2} |B_d|.$$

This covers the Marchioro-Pulvirenti estimate used for 2D-Euler (1985): $m = 2$, and $d = 2$, and for homogeneous steady states for VP systems.

This estimate was used by Caglioti and Rousset to study long time behavior of some N particles systems (2007): homogeneous steady states to regularized VP, Euler 2D.

Statement of stability inequalities for VP

The energy space

$$\mathcal{E} = \{f \in L^\infty : f \geq 0, \ \|(1+|v|^2)f\|_{L^1} < \infty\}.$$

Theorem: Quantitative stability (ML).

We have the following

i) There exist a constant $K_0 > 0$ depending only on f_0 such that or all $f \in \mathcal{E}$

$$\|f - f_0\|_{L^1} \leq \quad \|f^* - f_0^*\|_{L^1} + K_0\left[\mathcal{H}(f) - \mathcal{H}(f_0) + 2|\phi_{f_0}(0)|\|f^* - f_0^*\|_{L^1} + \|\nabla\phi_f - \nabla\phi_{f_0}\|_{L^2}^2\right]^{1/2}.$$

ii) There exist constants $K_0, R_0 > 0$ depending only on f_0 such that, for all $f \in \mathcal{E}$ satisfying

$$\inf_{z \in \mathbb{R}^3} \left(\|\phi_f - \phi_{f_0}(. - z)\|_{L^\infty} + \|\nabla\phi_f - \nabla\phi_{f_0}(. - z)\|_{L^2}\right) < R_0,$$

there holds:

$$\|f - f_0(. - z_{\phi_f})\|_{L^1} + \|\nabla\phi_f - \nabla\phi_{f_0}(. - z_{\phi_f})\|_{L^2} \leq \quad \|f^* - f_0^*\|_{L^1} + K_0\left[\mathcal{H}(f) - \mathcal{H}(f_0) + K_0\|f^* - f_0^*\|_{L^1}\right]^{1/2}.$$

Some perspectives

- Non decreazing steady states?
- Periodic domain in space: first non linear stability result for HMF (ML, A. M. Luz, F. Méhats, 2017).
- 2D Euler: similar structure as VP, but more difficult: partial result (ML, *2016.*)
- Vlasov-Einstein even in simplified geometries.
- Refined rearrangement inequalities: Riesz, Polya-Zgo ...
- Linear and non linear instabilities: strategy by Lin-Strauss for the linear case completed by a non linear iterative method (as in Han-Kwan and Hauray 2015-2016 for the homogenous steady states)