J.L. Lions' problem on the maximal regularity for non-autonomous equations

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Marrakech, April 2018

Autonomous Equations

Consider the Cauchy problem

$$\begin{cases} \partial_t u(t) + Au(t) = f(t), \quad t \in [0, T], \\ u(0) = 0. \end{cases} \tag{1}$$

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Maximal L^p-regularity:

$$f \in L^p(0, T, E) \Rightarrow \exists u \in W^{1,p}(0, T, E) \cap L^p(0, T, D(A)) \text{ satisfying } (1).$$

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$\Rightarrow$ An apriori estimate:

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Works by Da Prato-Grisvard, Dore-Venni, Lamberton, L. Weis, Kalton-Lancien, + . . . + . . . + . . .

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- L. Weis(2001): $E = L^q$, MR is equivalent to $\mathcal{R}$-boundedness of e^{-zA} (complex $z \in \Sigma_\theta$):

$$\int_0^1 \|\sum_{j=0}^N r_j(t) e^{-z_j A} f_j\|_E dt \leq C \int_0^1 \|\sum_{j=0}^N r_j(t) f_j\|_E dt \quad \forall f_j \in E, \forall z_j \in \Sigma_\theta$$

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- Kalton-Lancien(2000): "negative results".

Non-autonomous Equations

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Works by: H. Amann, M. Giga, Y. Giga, H. Sohr, Prüss-Schnaubelt, Arendt-Chill-Fornaro-Poupaud, Batty-Chill-Srivastava, ... assuming: $D(A(t)) = D(A(0)) = D$ + continuity of $t \to A(t)u$.

J.L. Lions' theorems

Assumptions-Notations: H, V Hilbert spaces, $V \subset H$ continuously and densely, and $\mathfrak{a}(t,\cdot,\cdot) : V \times V \to \mathbb{C}$ sesquilinear forms s.t. :
- $|\mathfrak{a}(t,u,v)| \leq M\|u\|_V \|v\|_V$, $u, v \in V, t \in [0,T]$;
- $\mathrm{Re}\,\mathfrak{a}(t,u,u) \geq \delta\|u\|_V^2 - k\|u\|_H^2$,
- $t \mapsto \mathfrak{a}(t,u,v)$ measurable for all $u, v \in V$.

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Example:

$$\mathfrak{a}(t, u, v) = \sum_{k,l} \int_\Omega a_{kl}(t, x) \partial_l u \partial_k v \, dx, \;\; W_0^{1,2}(\Omega) \subset V \subset W^{1,2}(\Omega)$$

$$A(t) = -\sum_{k,l} \partial_k (a_{kl}(t, x) \partial_l) + \text{boundary conditions given by}\, V.$$

- If $V = W_0^{1,2}(\Omega)$ then we have the Dirichlet boundary conditions.
- If $V = W^{1,2}(\Omega)$ then we have Neumann type boundary conditions.

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Theorem (J.L. Lions)

- If $t \mapsto \mathfrak{a}(t, u, v)$ is C^1 and $\mathfrak{a}(t, ., .)$ are symmetric then (NACP) with $u_0 = 0$ has maximal L^2-regularity in H.
- If $t \mapsto \mathfrak{a}(t, u, v)$ is C^2 and $\mathfrak{a}(t, ., .)$ are symmetric then (NACP) with $u_0 \in D(A(0))$ has maximal L^2-regularity in H.

J.L. Lions' problem (1961)

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Bardos (1971): $u_0 \in V$ is allowed provided $D(A(t)^{1/2}) = V$ and strong regularity of $A(t)$ with respect to t.

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Theorem (Ou-Spina, J.D.E 2010)

Suppose $t \mapsto \mathfrak{a}(t, u, v)$ is Hölder continuous in the sense: for some $\alpha > \frac{1}{2}$, $|\mathfrak{a}(t, u, v) - \mathfrak{a}(s, u, v)| \leq K|t - s|^{\alpha}\|u\|_V\|v\|_V$ for all $s,\ t \in [0, T]$ and $u,\ v \in V$. Then (NACP) has maximal L^p-regularity in H when $u_0 = 0$.

$\Rightarrow$ partial answer to Problem 1.

Theorem (Haak-Ou, Math. Ann. 2015)

Suppose that

$$|\mathfrak{a}(t,u,v) - \mathfrak{a}(s,u,v)| \leq \omega(|t-s|)\|u\|_V\|v\|_V$$

with $\omega : [0,T] \to [0,\infty)$ *a non-decreasing function such that*

$$\int_0^T \frac{\omega(t)}{t^{3/2}} dt < \infty.$$

Then the non autonomous Cauchy problem (NACP), with $u_0 = 0$*, has maximal* L^p*-regularity in* H *for all* $p \in (1,\infty)$*. If in addition* ω *satisfies*

$$\int_0^T \frac{\omega(t)^p}{t^{\frac{1+p}{2}}} dt < \infty$$

then (NACP) has maximal L^p*-regularity for all* $u_0 \in (H, D(A(0)))_{1-1/p,p}$*.*

In the particular case $p = 2$, maximal L^2-regularity holds for all $u_0 \in D(A(0)^{1/2})$ if $t \mapsto \mathfrak{a}(t,u,v)$ is piecewise α- Hölder continuous for some $\alpha > 1/2$. This gives a complete answer to Problem 2 by Lions.

Corollary

Suppose that the form $\mathfrak{a}$ is piecewise α-Hölder continuous for some $\alpha > 1/2$. That is, there exist $t_0 = 0 < t_1 < ... < t_k = \tau$ such that on each interval (t_i, t_{i+1}) the form is the restriction of a α–Hölder continuous form on $[t_i, t_{i+1}]$. Assume in addition that at the discontinuity points, we have $D((\delta + A(t_j^-))^{1/2}) = D((\delta + A(t_j^+))^{1/2})$. Then (NACP) has maximal L^2–regularity for all $u_0 \in D((\delta + A(0))^{1/2})$ and there exists a positive constant C such that

$$\|u\|_{W_2^1(0,\tau;H)} + \|A(\cdot)u(\cdot)\|_{L^2(0,\tau;H)} \leq C\left[\|f\|_{L^2(0,\tau;H)} + \|u_0\|_{D((\delta+A(0))^{1/2})}\right].$$

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The idea is to apply the previous theorem on each sub-interval (t_i, t_{i+1}), prove $u(t_{i+1}) \in D((\delta + A(t_j^-))^{1/2}) = D((\delta + A(t_j^+))^{1/2})$ and glue the corresponding solutions.

The result in the corollary does NOT hold if $D((\delta + A(t_j^-))^{1/2}) \neq D((\delta + A(t_j^+))^{1/2})$! Observation due to D. Dier.

Theorem (Ou., Arch. Math 2015)

Suppose that for some $\beta, \gamma \in [0,1]$

$$|\mathfrak{a}(t,u,v) - \mathfrak{a}(s,u,v)| \leq \omega(|t-s|)\|u\|_{[H,V]_\beta}\|v\|_{[H,V]_\gamma}$$

where $\omega : [0,T] \to [0,\infty)$ *is a non-decreasing function such that*

$$\int_0^T \frac{\omega(t)}{t^{1+\frac{\gamma}{2}}} dt < \infty.$$

Then (NACP) with $u_0 = 0$ *has maximal* L^p*-regularity in* H *for all* $p \in (1,\infty)$. *If in addition,*

$$\int_0^T \frac{w(t)^p}{t^{\frac{1}{2}(\beta+p\gamma)}} dt < \infty$$

then (NACP) has maximal L^p*-regularity in* H *for all* $u_0 \in (H, D(A(0)))_{1-\frac{1}{p},p}$.

A Related result by Arendt and Monniaux (Math. Nach. 2016)

S. Fackler's negative result:

Theorem (Fackler, AIHP 2016)

There exist (even symmetric) forms such that $t \mapsto \mathfrak{a}(t,u,v)$ is $\frac{1}{2}$-Hölder continuous such that the corresponding non-autonomous Cauchy problem (NACP) does NOT have maximal L^2-regularity.

This proves that for the remaining part C^α for $\alpha \leq 1/2$ in Lions' problem (Problem 1 above) the answer is no in general. In particular, our previous result with $t \mapsto \mathfrak{a}(t,u,v)$ is piecewise C^α for some $\alpha > 1/2$ is sharp.

Theorem (Achache-Ou, Studia Math. 2018)

Suppose that for some $\beta, \gamma \in [0,1]$

$$|\mathfrak{a}(t,u,v) - \mathfrak{a}(s,u,v)| \le \omega(|t-s|)\|u\|_{[H,V]_\beta}\|v\|_{[H,V]_\gamma},$$

where $\omega : [0,T] \to [0,\infty)$ *is a non-decreasing function such that :*

$$\int_0^T \frac{\omega(t)}{t^{1+\frac{\gamma}{2}}}\,dt < \infty.$$

Let $B(t), P(t)$ *be bounded operators on* H *such that* $t \mapsto B(t)$ *is continuous on* $[0,T]$ *with values in* $\mathcal{L}(H)$ *and* $Re(B(t)^{-1}x, x) \ge \delta\|x\|_H^2$*. Then the Cauchy problem*

$$u'(t) + B(t)A(t)u(t) + P(t)u(t) = f(t), u(0) = 0$$

has maximal L^p*-regularity in* H *for all* $p \in (1,\infty)$*. If in addition,*

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then: $u'(t) + B(t)A(t)u(t) + P(t)u(t) = f(t), u(0) = u_0$
has maximal L^p*-regularity in* H *provided* $u_0 \in (H, D(A(0)))_{1-\frac{1}{p},p}$*.*

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Then the solution $u(t) \in D(\mathcal{A}(t))$ exists in V' by Lions' theorem.

In addition $\mathcal{A}(t)u(t) = (Q\mathcal{A}(\cdot)u(\cdot))(t) + (Lf)(t) + (Ru_0)(t)$,

where

$(Qg)(t) := \int_0^t \mathcal{A}(t)e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t) - \mathcal{A}(s))\, \mathcal{A}(s)^{-1} g(s)\, ds$

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$$\begin{aligned} &\|\mathcal{A}(t)e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t) - \mathcal{A}(s))\,\mathcal{A}(s)^{-1}g(s)\|_H \\ &\le \|\mathcal{A}(t)e^{-(t-s)\mathcal{A}(t)}\|_{V'-H}\,\|\mathcal{A}(t) - \mathcal{A}(s)\|_{V-V'}\,\|\mathcal{A}(s)^{-1}g(s)\|_V \\ &\le C(t-s)^{-3/2}\omega(t-s)\|g(s)\|_H. \end{aligned}$$

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L is a pseudo-differential operator

$$Lf(t) = \mathcal{F}^{-1}\big(\xi \mapsto \sigma(t,\xi)\mathcal{F}(\xi)\big),$$

with operator-valued symbol

$$\sigma(t,\xi) = \begin{cases} A(0)(i\xi + A(0))^{-1} & \text{if} \quad t < 0 \\ A(t)(i\xi + A(t))^{-1} & \text{if} \quad 0 \le t \le \eta \\ A(\eta)(i\xi + A(\eta))^{-1} & \text{if} \quad t \ge \eta \end{cases}$$

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Theorem

Let T be a pseudo-differential operator with symbol $\sigma(t, \xi)$. Suppose that there exists a non-decreasing function $w : [0, \infty) \to [0, \infty)$ such that

$$\|\partial_\xi^\alpha \sigma(x, \xi)\|_{\mathcal{L}(H)} \leq C_\alpha < \xi >^{-|\alpha|}$$

$$\|\partial_\xi^\alpha \sigma(x, \xi) - \partial_\xi^\alpha \sigma(x', \xi)\|_{\mathcal{L}(H)} \leq C_\alpha < \xi >^{-|\alpha|} \omega(|x - x'|)$$

for all $|\alpha| \leq [\frac{n}{2}] + 2$ and some positive constant C_α. Suppose in addition that

$$\int_0^1 \omega(t)^2 \, \frac{dt}{t} < \infty,$$

then T is a bounded operator on $L^2(\mathbb{R}; H)$.

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$$A(t)e^{-tA(t)}u_0 - A(0)e^{-tA(0)}u_0 = \frac{1}{2\pi i} \int_\Gamma z e^{-tz} \left[R(z, A(t)) - R(z, A(0)) \right] dz$$

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Problem: What about the case $p = 2$, i.e., $\mathcal{A}(.) \in H^{\frac{1}{2}}(0, \tau; \mathcal{L}(V, V'))$?

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- The order of smoothness $\frac{1}{2}$ can be improved into $\frac{\gamma}{2}$ if in addition

$$\mathcal{A}(t) - \mathcal{A}(s) : V \to [H, V]_\gamma$$

This latter condition holds in some situation such as Robin boundary conditions or Schrödinger operators with time dependent potentials.

Examples:

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $H = L^2(\Omega)$.

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A Linear Problem (1):

$$\begin{cases} \partial_t u(t) - \sum_{k,l=1}^{d} \partial_k (a_{kl}(t,x)\partial_l u) = f(t), t \in [0,T], \\ u(0) = u_0 \in V \\ + \text{Dirichlet or Neumann on b.c. on } \partial\Omega. \end{cases}$$

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Proof: apply the above results to

$$\mathfrak{a}(t,u,v) = \sum_{k,l} \int_\Omega a_{kl}(t,x)\partial_l u \partial_k v \, dx$$

and note that $D(A(t)^{1/2}) = W_0^{1,2}(\Omega)$ or $W^{1,2}(\Omega)$ depending on the b.c. This is the Kato square root property (cf. Auscher, Hofmann, Lacey, McIntosh and Tchamitchian 2002).

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Indeed,

$$\begin{aligned} &|\mathfrak{a}(t;u,v) - \mathfrak{a}(s;u,v)| \\ &= |\int_{\partial\Omega} [\beta(t,.) - \beta(s,.)] \mathrm{Tr}(u) \mathrm{Tr}(v)\, \mathrm{d}\sigma| \\ &\le C|t-s|^\alpha \left(\|u\|_{H^{1/2+\varepsilon}(\Omega)} \|v\|_{H^{1/2+\varepsilon}(\Omega)} \right), \end{aligned}$$

(the trace operator is bounded from $H^{1/2+\varepsilon}(\Omega)$ into $L^2(\partial\Omega)$ for $\varepsilon > 0$).

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$u \in W^{1,2}(0,T,L^2(\Omega)) \cap L^2(0,T,W^{1,2}(\Omega))$ such that $Su = u$.

Two remaining problems:

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- What about the theorem with condition $H^{1/2}$ if the uniform Kato square root property is not satisfied (or not known) ?

- For the particular case of divergence form elliptic operators

$$A(t) = -\sum_{k,j} \partial_k(a_{kj}(t,.)\partial_j)$$

can one relax the required regularity $t \mapsto a_{kj}(t,x)$ is $C^{1/2+\varepsilon}$ (or $H^{1/2}$) and obtain maximal regularity ?

Thank you for your attention !!