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FRACTIONAL KARDAR-PARISI-ZHANG EQUATION

Boumediène Abdellaoui

University of Tlemcen, Algeria.

This talk is based on the work,

B. Abdellaoui, I. Peral, *Towards a deterministic KPZ equation with fractional diffusion: The Stationary case*, Nonlinearity 2018

B. Abdellaoui, B. Ochoa, I. Peral, *Nonlocal elliptic problem with an absorption term depending on the gradinet*, Preprint 2018

Presentation and Motivations

In this talk we analyze the existence and regularity of positive solution to problem

$$(Q)\begin{cases} (-\Delta)^s u = |\nabla u|^q + c\,f(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $f \geq 0$, $c \geq 0, 1 < q$, $s \in (0,1)$ and

$$(-\Delta)^s u(x) = \text{ P.V. } \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy.$$

- If $s = 1$, problem (Q) is the stationary part of the equation

$$u_t - \epsilon \Delta u = |\nabla u|^2,$$

which may be viewed as the viscosity approximation as $\epsilon \to 0^+$ of Hamilton-Jacobi type equations from stochastic control theory. See for instance,

P.L. Lions, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations: part 1,2: Viscosity solutions and uniqueness*. Comm. PDE 8 (1983).

- The same parabolic equation appears in the physical theory of growth and roughening of surfaces, where it is known as the Kardar-Parisi-Zhang equation. See for instance,

M. Kardar, G. Parisi, Y.C. Zhang, Phys. Rev. Lett. 56, (1986), 889-892.

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- If $s \in (0,1)$, we have the next "nonlocal version" of the Burgers equation

$$v_t = -a(-\Delta)^s v + b(x,t)\nabla v^r \text{ in } \mathbb{R}^N \times (0,T).$$

See for instance,

W. A. Woyczyński, *Burgers-KPZ turbulence*. Göttingen lectures. Lecture Notes in Mathematics, 1700. Springer-Verlag, Berlin, 1998.

Some precedent works: local case:

- **J. Leray and J.-L. Lions**, Bull. Soc. Math. France **93**, (1965), 97-107.
- **G. Stampacchia**, *Equations elliptiques du second ordre à coefficients discontinus.* Les Presses de lUniversité de Montréal, Montréal, 1966.
- **L. Boccardo, F. Murat, J.-P. Puel**, Portugal Math. **41** (1982), 507-534.
- **J. M. Lasry, P. L. Lions**, Math. Ann. 283, 583-630 (1989).
- **L. Boccardo, T. Gallouët,** J. Funct. Anal. 87 (1) (1989) 149169.
- **N.E. Alaa, M. Pierre**, SIAM J. Math. Anal., **24**, (1993), 23-35.
- **D. Giachetti, F. Murat**, Boll. Unione Mat. Ital. (9) 2 (2009), no. 2, 349-370.

Some precedent works: Non local case:

- **K. Bogdan, T. Jakubowski**, Potential Anal 36 (2012), 455-481.
- **H. Chen, L. Veron**, *Semilinear fractional elliptic equations involving measures*, J. Differential Equations 257 (2014) 1457-1486.
- **H. Chen, L. Veron**, *Semilinear fractional elliptic equations with gradient nonlinearity involving measures.* Journal of Functional Analysis 266 (2014) 5467-5492.
- **W. A. Woyczyński**, *Burgers-KPZ turbulence.* Göttingen lectures. Lecture Notes in Mathematics, 1700. Springer-Verlag, Berlin, 1998

Planning of the talk.

1. Preliminaries and auxiliary results.

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1. Preliminaries and auxiliary results.
2. Comparison principle and application.
3. Existence results via comparison principle: $q < 2s$.
4. Existence results via fixed point Theorem: $q \geq 2s$.
5. Further results and open problems

Preliminaries and auxiliary results.

In this section we present some useful results about the problem

$$(G)\begin{cases}(-\Delta)^s v = \nu & \text{in } \Omega,\\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega,\end{cases}$$

where ν is a bounded Radon measure.

Definition

We say that v is a weak solution to problem (Q) if $v \in L^1(\Omega)$, and for all $\phi \in \mathbb{X}_s$, we have

$$\int_\Omega v(-\Delta)^s \phi dx = \int_\Omega \phi d\nu,$$

where

$$\mathbb{X}_s \equiv \Big\{\phi \in \mathcal{C}(\mathbb{R}^{\mathrm{N}}) \text{ and } |(-\Delta)^s \phi(x)| < C \text{ in } \Omega\Big\}.$$

Using Approximation and duality arguments we get

Theorem

Let $f \in L^1(\Omega)$, then problem (G) has a unique weak solution $v = \lim_{n\to\infty} v_n$, where v_n solves

$$\begin{cases} (-\Delta)^s v_n = f_n(x) & \text{in } \Omega, \\ v_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

with $f_n = T_n(f)$. Moreover, $v \in L^q(\Omega), \forall q < \frac{N}{N-2s}$ and

$$T_k(v_n) \to T_k(v) \text{ strongly in } H_0^s(\Omega), \quad \forall k > 0,$$

$$\left|(-\Delta)^{\frac{s}{2}} v\right| \in L^r, \qquad \forall\, r \in \left(1, \frac{N}{N-s}\right).$$

See for instance,

Leonori-Peral-Primo-Soria, Discrete and Continuous Dynamical Systems- A, 35 (2015).

Abdellaoui-Attar-Bentifour, Advanced Nonlinear Analysis (2016).

Assume that $\mu = \delta_{x_0}$, and let $\mathcal{G}_s$ be the Green kernel of $(-\Delta)^s$. Using a probabilistic approach we have the next estimate on $\mathcal{G}_s$:

Lemma

Assume that $s \in (\frac{1}{2}, 1)$, then

$$\mathcal{G}_s(x,y) \le C_1 \min\{\frac{1}{|x-y|^{N-2s}}, \frac{d^s(x)}{|x-y|^{N-s}}, \frac{d^s(y)}{|x-y|^{N-s}}\}, \qquad (1)$$

and

$$|\nabla_x \mathcal{G}_s(x,y)| \le C_2 \mathcal{G}_s(x,y) \max\{\frac{1}{|x-y|}, \frac{1}{d(x)}\}. \qquad (2)$$

In particular

$$|\nabla_x \mathcal{G}_s(x,y)| \le \frac{C}{|x-y|^{N-2s+1}}.$$

See for instance,

K. Bogdan, T. Kulczycki, A. Nowak, Illinois J. Math. 46 (2002) no 2, 541-556 . 1-144.

As a consequence of the previous estimates we get

Theorem

Suppose that $s \in (\frac{1}{2}, 1)$ and $f \in L^1(\Omega)$. The problem (G) has a unique weak solution in the sense of Definition 1 such that,

1. $|\nabla v| \in M^{p_*}(\Omega)$, *the Marcinkiewicz space, with* $p_* = \frac{N}{N-2s+1}$, *thus* $v \in W_0^{1,q}(\Omega)$ *for all* $q < p_*$. *Moreover*

$$||v||_{W_0^{1,q}(\Omega)} \leq C(N, q, \Omega)||f||_{L^1(\Omega)}.$$

2. $T_k(v) \in W_0^{1,\alpha}(\Omega) \cap H_0^s(\Omega)$ *for any* $1 < \alpha < 2s$ *and*

$$\int_\Omega |\nabla T_k(v)|^\alpha \leq Ck^{\alpha-1}||f||_{L^1(\Omega)}.$$

3. *For* $f \in L^1(\Omega)$, *setting* $T : L^1(\Omega) \to W_0^{1,q}(\Omega)$, *with* $T(f) = v$, *then* T *is a compact operator.*

(Main Lemma)

In general using Calderon-Zygmund theory we are able to prove

Lemma

(Main Lemma) Suppose that $f \in L^m(\Omega)$ with $m \geq 1$, then for all $p < \frac{mN}{N-m(2s-1)}$, there exists a positive constant $C \equiv C(\Omega, N, s, p)$ such that

$$||\nabla v||_{L^p(\Omega)} \leq C||f||_{L^m(\Omega)}. \tag{3}$$

Moreover,

1. *If $m = \frac{N}{2s-1}$, then $|\nabla v| \in L^p(\Omega)$ for all $p < \infty$.*
2. *If $m > \frac{N}{2s-1}$, then $v \in \mathcal{C}^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$.*

Remark If $f \in L^1(\Omega, d^\beta)$ with $0 \leq \beta \leq s$, then for

$$p \in (1, \frac{N}{N+\beta-2s}),$$

$$||v||_{W^{2s-\gamma,p}} \leq c(p,\Omega)||f||_{L^1(\Omega,d^\beta)},$$

where $\gamma = \beta + \frac{N}{p'}$ if $\beta > 0$ and $\gamma > \frac{N}{p'}$ if $\beta = 0$.

Comparison principle and applications

We need some results on problems with first order term. In first place the following Harnack inequality.

Theorem

(Harnack inequality) Assume that $B \in (L^{\sigma}(\Omega))^N$ *with* $\sigma > \frac{N}{2s-1}$ *and let* $w \in \mathcal{C}^{1,\alpha}(\Omega)$ *be a nonnegative function in* $\mathbb{R}^N$ *such that*

$$(-\Delta)^s w - \langle B(x), \nabla w \rangle = 0 \text{ in } \Omega.$$

Then for all $B_R \subset \Omega$*, with* $B_{2R} \subset \Omega$*, we have*

$$\sup_{B_R} w \le C \inf_{B_R} w$$

See for instance,

K. Bogdan, T. Jakubowski, Potential Anal 36 (2012).

We have the next Main comparison principle.

Theorem

Let $0 \leq g \in L^1(\Omega)$. Assume that for all $\xi_1, \xi_2 \in \mathbb{R}^N$,

$$H : \Omega \times \mathbb{R}^N \to \mathbb{R}^+ \text{ verifies } |H(x,\xi_1) - H(x,\xi_2)| \leq b(x)|\xi_1 - \xi_2|$$

where $b \in L^\sigma(\Omega)$ for some $\sigma > \frac{N}{2s-1}$. Consider w_1, w_2 two positive functions such that $w_1, w_2 \in W^{1,p}(\Omega)$ for all $p < p_*$, $(-\Delta)^s w_1, (-\Delta)^s w_2 \in L^1(\Omega)$, $w_1 \leq w_2$ in $\mathbb{R}^N \setminus \Omega$ and

$$(-\Delta)^s w_1 \leq H(x, \nabla w_1) + g \text{ in } \Omega,$$

$$(-\Delta)^s w_2 \geq H(x, \nabla w_2) + g \text{ in } \Omega.$$

Then, $w_2 \geq w_1$ in Ω.

To prove The Comparison Principle we need several auxiliary results.

Let us begin by the next uniqueness result.

Lemma

Let B be a vector field in Ω. Assume that $B \in (L^\sigma(\Omega))^N$ with $\sigma > \frac{N}{2s-1}$ and let w be a solution to the problem

$$\begin{cases} (-\Delta)^s w = \langle B(x), \nabla w\rangle & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4}$$

with $|\nabla w| \in M^{p_*}(\Omega)$, then $w = 0$.

Proof: We claim that $w \in \mathcal{C}^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$.

FIRST STEP. If $|\nabla w| \in L^m(\Omega)$ with $m = \frac{N\sigma}{(2s-1)\sigma - N}$, then

$$h(x) = |\langle B(x), \nabla w\rangle|, h \in L^{\frac{N}{2s-1}}(\Omega).$$

Main Lemma $\Rightarrow |\nabla w| \in L^p(\Omega)$ for all $p < \infty$. Thus $h \in L^\sigma(\Omega)$. Since $\sigma > \frac{N}{2s-1}$, then Main Lemma $\Rightarrow |\nabla w| \in \mathcal{C}^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1)$.

SECOND STEP. Wave just to show that $|\nabla w| \in L^{\frac{N\sigma}{(2s-1)\sigma - N}}(\Omega)$. This follows using: $|\nabla w| \in M^{p_*}(\Omega)$ and bootstrapping argument.

Let prove that $w \leq 0$.
By contradiction, if $C = \max_{x\in\Omega} w(x) > 0$, then $\exists x_0 \in \Omega$ such that $w(x_0) = C$.
Define $w_1 = C - w$, then $w_1 \geq 0$ in $\mathbb{R}^N$, $w_1(x_0) = 0$ and

$$(-\Delta)^s w_1 - B(x)\nabla w_1 = 0 \text{ in } \Omega.$$

Consider $B_R = B_r(x_0) \subset\subset \Omega$.

$$\text{Harnack inequality} \Rightarrow \sup_{B_r(x_0)} w_1 \leq C \inf_{B_r(x_0)} w_1 = 0.$$

Thus $w_1 \equiv 0$ in $B_r(x_0)$. Since Ω is a bounded domain, then, applying Harnack inequality a finite number of steps, we prove that $w_1 = 0$ in Ω. Thus $C \leq 0$ and then $w \leq 0$.
By linearity of the problem, $-w = 0$.
Thus $w \equiv 0$ and the result follows. ■

As a consequence we get the next existence and uniqueness result.

Lemma

Assume that $B \in (L^{\sigma_1}(\Omega))^N$ *with* $\sigma_1 > \frac{N}{2s-1}$ *and* $f \in L^2(\Omega)$. *The problem*

$$\begin{cases} (-\Delta)^s w = \langle B(x), \nabla w\rangle + f & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5}$$

has a unique solution $0 \le w \in H_0^s(\Omega) \cap W_0^{1,a}(\Omega)$ *where* $a = \frac{2N}{N-2(2s-1)}$.

1. *If* $f \in L^{\sigma_2}(\Omega$ *with* $\sigma_2 > \frac{N}{2s-1}$, *then* $w \in \mathcal{C}^{1,\alpha}(\Omega)$ *for some* $\alpha \in (0,1)$.

The dual problem

Recall that $s > \frac{1}{2}$. Define the space $E = W^{2s,2}(\Omega) \cap H_0^s(\Omega)$.

If $w \in E$, then $|\nabla w| \in L^{\frac{2N}{N-2(2s-1)}}(\Omega)$, since $B \in (L^{\sigma_1}(\Omega))^N$ with $\sigma_1 > \frac{N}{2s-1}$, then

$$|\langle B(x), \nabla w\rangle| \in L^2(\Omega) \ \forall w \in E.$$

Define $L(w) = (-\Delta)^s w - \langle B(x), \nabla w\rangle$, then $L : L^2(\Omega) \to L^2(\Omega)$, with $Dom(L) = E$.

It is clear that $Ker(L) = \{0\}$.

The dual problem

Now, for $u, v \in E$, we have

$$\langle L(u), v\rangle = \int_\Omega u\Big((-\Delta)^s v - \text{div}(B(x)v)\Big) u dx.$$

Define

$$K(v) = (-\Delta)^s v - \text{div}(B(x)v), \text{the adjoint of } L,$$

then

$$\langle K(v), u\rangle = \langle v, Lu\rangle.$$

Since dim Ker$(L) =$ dim Ker(K), Ker$(K) = \{0\}$. By the Fredholm alternative we reach that for all $f \in L^2(\Omega)$, the problem

$$\begin{cases} (-\Delta)^s v - \text{div}(B(x)v) &= f \quad \text{in } \Omega, \\ v &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

has a unique solution $u \in E$. Moreover if $f \geq 0 \Rightarrow u \geq 0$.

Proof of the comparison principle

We are now able to prove the Main comparison principle.
Let $w = w_1 - w_2$. We have just to show that $w^+ = 0$.

$$w \in W^{1,p}(\Omega)\, \forall p < p_* \,, (-\Delta)^s w \in L^1(\Omega),\; w \geq 0 \text{ in } \mathbb{R}^N \setminus \Omega.$$

and

$$(-\Delta)^s w \quad \leq \quad H(x, \nabla w_1) - H(x, \nabla w_2) \leq b(x)|\nabla w|.$$

Now, using Kato's inequality, we get

$$(-\Delta)^s w_+ \leq b(x)|\nabla w_+|, \qquad w_+ \in W_0^{1,q}(\Omega) \text{ for all } q < p_*. \quad (6)$$

Let v be the unique positive bounded solution to problem

$$\begin{cases} (-\Delta)^s v + \operatorname{div}(\mathcal{F}(x)v) &= 1 \quad \text{in } \Omega, \\ v &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{7}$$

where

$$\mathcal{F}(x) = \begin{cases} b(x)\dfrac{\nabla w_+(x)}{|\nabla w_+(x)|} & \text{if } |\nabla w_+(x)| \neq 0 \\ 0 & \text{in } |\nabla w_+(x)| = 0. \end{cases}$$

Taking v as a test a test function in (6), it follows that

$$\int_\Omega w_+ = \int_\Omega v(-\Delta^s)w_+ - \int_\Omega v\mathcal{F}\nabla w_+ \leq \\ \int_\Omega b(x)v(x)|\nabla w_+| - \int_\Omega b(x)|\nabla w_+(x)|v(x) = 0,$$

then it follows that $\int_\Omega w_+ \leq 0$. Thus $w \leq 0$ in Ω and we conclude. $\square$

As a byproduct of the previous result we obtain the following existence and uniqueness results.

Corollary

Let $g \in L^1(\Omega)$ be a nonnegative function. Suppose that $q \geq 1$ and $a > 0$, then the problem

$$\begin{cases} (-\Delta)^s w = \dfrac{|\nabla w|^q}{a + |\nabla w|^q} + g & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

has a unique nonnegative solution w such that $w \in W_0^{1,p}(\Omega)$ for all $p < p_*$ and $T_k(w) \in H_0^s(\Omega) \cap W_0^{1,\alpha}(\Omega)$ $\forall \alpha < 2s$.

Corollary

Consider the problem

$$\begin{cases} (-\Delta)^s w = |\nabla w|^q + \lambda g & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{8}$$

with $1 < q < p_*$ *and* $g \in L^1(\Omega)$, $g \geq 0$. *Then there exist* λ^* *such if* $\lambda < \lambda^*$, *problem (8) has a unique positive solution* w *such that* $w \in W_0^{1,p}(\Omega)$ *for all* $p < p_*$ *and* $T_k(w) \in H_0^s(\Omega)$ *for all* $k > 0$.

Proof The existence and regularity follow by the result of

Chen, Veron, J. Differential Equations 257 (2014) 1457-1486.

We prove the uniqueness.
If w_1 and w_2 are two positive solution to problem (8), setting $\bar{w} = w_1 - w_2$, then $\tilde{w} \in W_0^{1,p}(\Omega)$ for all $p < p_*$ and

$$(-\Delta)^s \bar{w} \leq b(x)|\nabla \bar{w}|,$$

where $b(x) = q\Big(|\nabla w_1| + |\nabla w_2|\Big)^{q-1}$.
$q < p_* \Rightarrow b \in L^\sigma(\Omega)$ for some $\sigma > \frac{N}{2s-1}$.
The comparison principle $\Rightarrow \bar{w}_+ = 0$.
In the same way and setting $\widehat{w} = w_2 - w_1$, we obtain that $\widehat{w}_+ = 0$.
Thus $w_1 = w_2$. ∎

Recall that we are considering the problem

$$(Q)\begin{cases} (-\Delta)^s u &= |\nabla u|^q + \lambda f & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u &> 0 & \text{in } \Omega, \end{cases}$$

where $s \in (\frac{1}{2}, 1)$, $q > 1$ and $f \in L^\sigma(\Omega)$ for some convenient $\sigma > 1$.

Thus we have the next existence result.

Theorem

Assume $f \in L^\infty(\Omega)$. Let w be a bounded supersolution to (Q) such that $w \in W^{1,q}(\Omega) \cap L^\infty(\Omega)$. Then problem (Q) has a minimal solution u such that $u \in W_0^{1,\alpha}(\Omega)$ for all $\alpha < 2s$.

We will use the next classical next.

Lemma

Let $0 < \mu < N$, $1 \le p < l < \infty$ be such that $\frac{1}{l} + 1 = \frac{1}{p} + \frac{\mu}{N}$. For $f \in L^p(\mathbb{R}^N)$, we define $J_\lambda(f)(x) = \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^\mu} dy$. Then

a) *If $p > 1$, then $||J_\lambda(f)||_l \le c_{p,q} ||f||_p$.*

b) *If $p = 1$, then $|\{x \in \mathbb{R}^N \,|\, J_\lambda(f)(x) > \sigma\}| \le \left(\frac{A||f||_1}{\sigma}\right)^{l_1}$ with $l_1 = \frac{N}{N-\mu}$.*

Proof: Let u_n be the unique solution to the approximating problem

$$(APR)\begin{cases} (-\Delta)^s u_n = \dfrac{|\nabla u_n|^q}{\frac{1}{n}+|\nabla u_n|^q} + \lambda f & \text{in } \Omega, \\ u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

The Comparison principle $\Rightarrow u_n \leq u_{n+1} \leq w$ for all n.
Hence, there exists u such that $u_n \uparrow u$ strongly in $L^a(\Omega)$ for all $a \geq 1$.

Let $g_n(x) = \dfrac{|\nabla u_n|^q}{\frac{1}{n}+|\nabla u_n|^q} + \lambda f$

We have that

$$u_n(x) = \int_\Omega \mathcal{G}_s(x,y) g_n(y)(y) dy, |\nabla u_n(x)| \leq \int_\Omega |\nabla_x \mathcal{G}_s(x,y)| g_n(y) dy.$$

Fix $1 < \alpha < 2s$, then

$$|\nabla u_n(x)|^\alpha \leq \Big(\int_\Omega |\nabla_x \mathcal{G}_s(x,y)| g_n(y) dy \Big)^\alpha$$

$$\leq \Big(\int_\Omega \frac{|\nabla_x \mathcal{G}_s(x,y)|}{\mathcal{G}_s(x,y)} \, \mathcal{G}_s(x,y) g_n(y) dy \Big)^\alpha$$

$$\leq \int_\Omega (\frac{|\nabla_x \mathcal{G}_s(x,y)|}{\mathcal{G}_s(x,y)})^\alpha dy \Big(\int_\Omega \mathcal{G}_s(x,y) g_n(y) dy \Big)^{\alpha-1}$$

$$\leq (\int_\Omega (\frac{|\nabla_x \mathcal{G}_s(x,y)|}{\mathcal{G}_s(x,y)})^\alpha dy) u_n^{\alpha-1}$$

Define

$$h(x,y) = \max\left\{\frac{1}{|x-y|}, \frac{1}{d(x)}\right\} \geq \frac{|\nabla_x \mathcal{G}_s(x,y)|}{\mathcal{G}_s(x,y)},$$

then

$$\int_\Omega |\nabla u_n|^\alpha dx \leq \int_\Omega |\nabla u_n(y)|^q \Big(\int_\Omega h^\alpha(x,y)\mathcal{G}_s(x,y) w^{\alpha-1}(x)\Big) dy$$
$$+\lambda \int_\Omega f(y) \Big(\int_\Omega h^\alpha(x,y)\mathcal{G}_s(x,y) w^{\alpha-1}(x) dx\Big) dy \equiv J_1 + J_2.$$

Since $w \in L^\infty(\Omega)$, $\alpha < 2s$ and the regularity of h, it holds

$$J_1 \leq C \int_\Omega |\nabla u_n(y)|^q \Big(\int_\Omega h^\alpha(x,y)\mathcal{G}_s(x,y) dx\Big) dy \leq C \int_\Omega |\nabla u_n(y)|^q dy$$

and

$$J_2 \leq C \int_\Omega f(y) dy$$

Therefore we conclude that (for all $\alpha < 2s$):

$$\int_\Omega |\nabla u_n(x)|^\alpha dx \leq C_1 \int_\Omega |\nabla u_n(x)|^q dx + C_2.$$

1. Thus $\{g_n\}_n$ is bounded in $L^{1+\epsilon}(\Omega)$ for some $\epsilon > 0$.
2. Up to a subsequence, $u_n \to u$ strongly in $W_0^{1,r}(\Omega)$ for all $r < p_*$ and $|\nabla u_n| \to |\nabla u|$ a. e. in Ω.
3. Vitali lemma $\Rightarrow u_n \to u$ strongly in $W_0^{1,\alpha}(\Omega)$ for all $\alpha < 2s$, in particular, for $\alpha = q$.

Thus u is a solution to (Q) with $u \in W_0^{1,\alpha}(\Omega)$ for all $a < 2s$. ∎

Now, as in the local case, we assume that $f \in L^{\gamma}(\Omega)$ for some $\gamma > \dfrac{N}{q'(2s-1)}$, $q' = \dfrac{q}{q-1}$. In order to obtain a solution, we need some extra condition on the supersolution. We obtain the following result.

Theorem

Assume that $f \in L^{\gamma}(\Omega)$ for some $\gamma > \frac{N}{q'(2s-1)}$. Let w be a nonnegative supersolution to (Q) such that $w \in W^{1,\alpha}(\Omega)$ for some $q < \alpha \leq 2s$. Suppose that the following estimate holds,

$$\sup_{x\in\Omega} \int_{\Omega} \frac{w^{\alpha-1}(x)\mathcal{G}_s(x,y)}{|x-y|^{\alpha}} dx \leq C. \tag{9}$$

Then problem (Q) has a solution u such that $u \in W_0^{1,\alpha}(\Omega)$ and $T_k(u) \in H_0^s(\Omega) \cap W_0^{1,\sigma}(\Omega)$ for all $\sigma < 2s$.

As a direct application of the previous Theorem, we get the following application to a concrete case.

Theorem

Assume that $q > p_*$ *and let* $f(x) = \dfrac{1}{|x|^\theta}$ *with* $0 < \theta < q'(2s-1)$*.*
Then there exists λ^* *such that for all* $\lambda < \lambda^*$*, problem (Q) has a solution* u *with* $u \in W_0^{1,\alpha_0}(\Omega)$ *for* $q < \alpha_0 < \min\{\frac{N}{\theta-2s+1}, 2s\}$ *and* $T_k(u) \in H_0^s(\Omega) \cap W_0^{1,\sigma}(\Omega)$ *for all* $\sigma < 2s$*..*

Remarks

We do not reach the case $q \geq 2s$:

1. In this case there are difficulties to use the comparison arguments. It is possible to find a supersolution but it is not clear how to pass to the limit in the gradient term when dealing with the family of approximating problems.

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1. In this case there are difficulties to use the comparison arguments. It is possible to find a supersolution but it is not clear how to pass to the limit in the gradient term when dealing with the family of approximating problems.
2. If $q = 2s$, in the local case this difficulty is overpasses by using convenient nonlinear test functions and a suitable change of variable.
3. In the nonlocal framework it seems to be necessary to change this point of view and to adapt new approach.

The critical case $q = 2s$

Using in a convenient way the Schauder fixed point theorem we get the next existence result.

Theorem

Suppose that Ω is a bounded regular domain and that $f \in L^m(\Omega)$ where $m > \frac{N}{2s}$. Then there exists $\lambda^*(f) > 0$ such that for all $\lambda < \lambda^*$, problem (Q) has a solution $u \in W_0^{1,2s}(\Omega)$.

For the local case, see for instance,

V.G. Maz'ya, I.E. Verbitsky, Ark. Mat. 33 (1995).

T. Mengesha, P. Nguyen Cong, J. Differential Equations 260 (2016), no. 6.

The main tool of the proof will be the next Calderon-Zygmund type results.

Lemma

(Main Lemma) Suppose that $f \in L^m(\Omega)$ *with* $m \geq 1$*, then for all* $p < \frac{mN}{N-m(2s-1)}$*, there exists a positive constant* $C \equiv C(\Omega, N, s, p)$ *such that*

$$||\nabla v||_{L^p(\Omega)} \leq C||f||_{L^m(\Omega)}. \tag{10}$$

Moreover,

1. *If* $m = \frac{N}{2s-1}$*, then* $|\nabla v| \in L^p(\Omega)$ *for all* $p < \infty$.
2. *If* $m > \frac{N}{2s-1}$*, then* $v \in \mathcal{C}^{1,\alpha}(\Omega)$ *for some* $\alpha \in (0, 1)$.

Proof:

Suppose that $f \in L^m(\Omega)$ where $m > \frac{N}{2s}$.

Since $2s > 1$, then $\exists \lambda^* > 0$ such that for some $l > 0$, we have

$$C_0(l + \lambda^* ||f||_{L^m(\Omega)}) = l^{\frac{1}{2s}}.$$

Fix $\lambda < \lambda^*$ and define the set

$$E = \{v \in W_0^{1,1}(\Omega) : v \in W_0^{1,2sm}(\Omega) \text{ and } ||\nabla v||_{L^{2sm}} \leq l^{\frac{1}{2s}}\}.$$

It is easy to check that E is a closed convex set of $W_0^{1,1}(\Omega)$.

Consider the operator

$$\begin{array}{rcl} T : E & \to & W_0^{1,1}(\Omega) \\ v & \to & T(v) = u \end{array}$$

where u is the unique solution to problem

$$\left\{ \begin{array}{rcll} (-\Delta)^s u & = & |\nabla v|^{2s} + \lambda f & \text{in } \Omega, \\ u & = & 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u & > & 0 & \text{in } \Omega. \end{array} \right.$$

Since $|\nabla v|^{2s} + \lambda f \in L^1(\Omega)$, then $|\nabla u| \in L^q(\Omega)$ for all $q < p_* = \frac{N}{N-2s+1}$. Hence T is well defined.

We claim that

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PROOF OF (1).
By The Main Lemma, since $\sigma_0 = 2sm < \frac{mN}{N-m(2s-1)}$, it follows that

$$||\nabla u||_{L^{\sigma_0}(\Omega)} \leq C_0 \left\| |\nabla v|^{2s} + \lambda f \right\|_{L^m(\Omega)}.$$

Thus

$$\begin{aligned} ||\nabla u||_{L^{\sigma_0}(\Omega)} &\leq C_0(||\nabla v||^{2s}_{L^{2sm}(\Omega)} + \lambda ||f||_{L^m(\Omega)}) \\ &\leq C_0(l + \lambda^* ||f||_{L^m(\Omega)}) = l^{\frac{1}{2s}}. \end{aligned}$$

Hence $u \in E$.

PROOF OF (2). Consider $\{v_n\}_n \subset E$ such that $v_n \to v$ strongly in $W_0^{1,1}(\Omega)$, define $u_n = T(v_n), u = T(v)$.
We have to show that $u_n \to u$ strongly in $W_0^{1,1}(\Omega)$. To do this we will prove that

$$||\nabla v_n - \nabla v||_{L^{2s}(\Omega)} \to 0 \text{ as } n \to \infty.$$

Since $\{v_n\}_n \subset E$ and $||v_n - v||_{W_0^{1,1}(\Omega)} \to 0$ as $n \to \infty$, then

$$\nabla v_n \to \nabla v \text{ strongly in } (L^1(\Omega))^N \text{ and } ||\nabla v_n||_{L^{2sm}(\Omega)} \leq C.$$

Since $2sm > 1$, setting $a = \frac{2s\delta}{2sm-1} < 1$ and by Hölder inequality,

$$||\nabla v_n - \nabla v||_{L^{2s}(\Omega)} \leq C||\nabla v_n - \nabla v||^{\frac{a}{2s}}_{L^1(\Omega)} \to 0 \text{ as } n \to \infty.$$

Now, by using the definition of u_n and u, there results that $u_n \to u$ strongly in $W_0^{1,1}(\Omega)$. Thus T is continuous.

PROOF OF (3): T is compact respect to the topology of $W_0^{1,1}(\Omega)$.
Let $\{v_n\}_n \subset E$ with $||v_n||_{W_0^{1,1}(\Omega)} \leq C$.
Since $\{v_n\}_n \subset E$, then $||\nabla v_n||_{L^{2sm}(\Omega)} \leq C$, up to a subsequence, $v_{n_k} \rightharpoonup v$ weakly in $W_0^{1,2sm}(\Omega)$.
Define

$$F_n = |\nabla v_n|^{2s} + \lambda f, F_n = |\nabla v|^{2s} + \lambda f,$$

then F_n is bounded in $L^{1+\delta}(\Omega)$ and $F_n \rightharpoonup F$ weakly in $L^{1+\delta}(\Omega)$.
The compactness result $\Rightarrow u_{n_k} \to u$ strongly in $W_0^{1,1}(\Omega)$, hence T compact.
As a conclusion and using the Schauder Fixed Point Theorem, there exists $u \in E$ such that $T(u) = u$, then $u \in W_0^{1,2sm}(\Omega)$ and u solves (Q). ■

Remarks

The solution obtained above is the unique solution in E.

Indeed, assume $u_1, u_2 \in E$ solutions to problem (Q).
Then $||\nabla u_1||_{L^{2sm}} < \infty$ and $||\nabla u_2||_{L^{2sm}} < \infty$.
Define $w = u_1 - u_2$, then $||\nabla w||_{L^{2sm}} < \infty$ and w solves

$$\begin{cases} (-\Delta)^s w = |\nabla u_1|^{2s} - |\nabla u_2|^{2s} & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Setting $b(x) = |\nabla u_1|^{2s-1} + |\nabla u_2|^{2s-1}$, then

$$(-\Delta)^s w \leq 2sb(x)|\nabla w| \text{ in } \Omega.$$

Since $m > \frac{N}{2s}$, then $b \in L^\sigma(\Omega)$ for $\sigma > \frac{N}{2s-1}$.
The comparison principle $\Rightarrow w_+ = 0$. Thus $u_1 \leq u_2$. In a similar way we get $u_2 \leq u_1$. Hence $u_1 = u_2$.

The supercritical case $q > 2s$

In a similar way we can handle the supercritical case $q > 2s$.

Theorem

Suppose that Ω is a bounded regular domain and that $f \in L^m(\Omega)$ where $m > \frac{N}{q'(2s-1)}$. Then there exists $\lambda^*(f) > 0$ such that for all $\lambda < \lambda^*$, problem (Q) has a solution $u \in W_0^{1,q}(\Omega)$.

Proof We chose $l > 0$ and σ_0 such that

$$\sigma_0 \equiv qm < \frac{mN}{N - m(2s-1)} \text{ and } C_0(l + \lambda^* ||f||_{L^m(\Omega)}) = l^{\frac{1}{q}}.$$

Define the set

$$E = \{v \in W_0^{1,1}(\Omega) : \text{ and } ||\nabla v||_{L^{qm}} \le l^{\frac{1}{q}}\}. \qquad (11)$$

Consider $T_q : E \to W_0^{1,1}(\Omega)$ defined by $u = T_q(v)$ where u is the unique solution to problem

$$\begin{cases} (-\Delta)^s u = |\nabla v|^q + \lambda f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u > 0 & \text{in } \Omega. \end{cases}$$

Fix $\lambda < \lambda^*(f)$, we can prove that T_q has a fixed point in E_q and then problem (Q) has a solution $u \in E_q$. ■

remark The solution obtained above is unique in E_q.

Remarks

For m fixed, the condition on q (that is $q < \frac{N}{N-m(2s-1)}$) is optimal. In the sense that if $q > \frac{N}{N-m(2s-1)}$, then there exists $f \in L^m(\Omega)$ such that the problem (Q) has non solution.

To see that, for $m = 1$, fix $q_1 > \frac{N}{N-(2s-1)}$.
Let $f(x) = \frac{1}{|x|^{N-\epsilon}}$ with $\Omega = B_R(0)$.
If there exists a solution, then $u \geq \frac{C}{|x|^{N-2s-\epsilon}} = w$ in $B_\eta(0)$.

$u \in W^{1,q_1}(B_\eta) \Rightarrow u \in L^{q_1^*}(B_\eta)$. Hence $w \in u \in L^{q_1^*}(B_\eta)$.

Thus $q_1^*(N - 2s - \epsilon) = \frac{q_1 N}{N-q_1}(N - 2s - \epsilon) < N$.

Since $q_1 > \frac{N}{N-(2s-1)}$, choosing ϵ small, we get $\frac{q_1 N}{N-q_1}(N - 2s - \epsilon) > N$. Contradiction.

Remarks

There exists $\lambda^* > 0$ such that if $\lambda > \lambda^*$, there non positive solution.

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There exists $\lambda^* > 0$ such that if $\lambda > \lambda^*$, there non positive solution.

$$\lambda^* = \inf_{\{\theta \in \mathcal{C}_0^\infty(\Omega), \theta \gneqq 0\}} \frac{\displaystyle\int_\Omega \phi_\theta |\nabla \psi_\theta|^q dx}{\displaystyle\int_\Omega f \, \phi_\theta},$$

where ϕ_θ satisfies

$$\begin{cases} (-\Delta^s)\phi_\theta &= \theta \quad \text{in } \Omega, \\ \phi_\theta &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and ψ_θ is the solution to the problem

$$\begin{cases} -\text{div}(\phi_\theta |\nabla \psi|^{q-2} \nabla \psi) &= \theta \quad \text{in } \Omega, \\ \psi &= 0 \quad \text{n } \partial\Omega. \end{cases}$$

Some open problems

- Regularity of the gradient without using representation formula:The regularity result proved in The main Lemma is the key in order to show the existence results and it depends directly on the representation formula and in the pointwise estimates on the Green function G_s.

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- Regularity of the gradient without using representation formula:The regularity result proved in The main Lemma is the key in order to show the existence results and it depends directly on the representation formula and in the pointwise estimates on the Green function G_s.
- In the local case $s = 1$, for $q = 2$, an exponential regularity is obtained for any non negative solution. Namely, any positive solution satisfies $e^{\alpha u} - 1 \in W_0^{1,2}(\Omega)$ for all $\alpha < \frac{1}{2}$. It seems to be natural to ask if any exponential regularity holds for positive solutions to problem (Q) with $q = 2s$.

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with $t < s$. Existence holds under suitable condition on f and λ. The question of uniqueness is still open.

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Thank you for your attention